On the implementation of Preference Relational Systems in a Computer Aided Decision Support System

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Abstract

Preference Systems are models used to manage the preferences on a set of alternatives. **Preference Relational Systems** are preference systems based on a series of exhaustive and mutually exclusive binary relations. Preferences are best shown about paired comparisons and **Binary Relations** are the most intuitive and suitable mathematical tool to modelize preferences based on paired comparisons. In addition to a **preference** relation P and a **indifference** relation I, a **general preference relational system** includes a **semipreference** relation S between the preference and a indifference and the **incomparability** relation X corresponding to the nonexistence of preference information to relate the alternatives. The system is **simple** when the semipreference is excluded and **complete** when the incomparability is excluded. The most important **structures** of these systems are introduced by **mixed transitivity** properties (conditions to infer a preference relation from a chain of related alternatives). These structures (the preorder, the semiorder, the interval order, the quasi-order, the double semiorder and the pseudo-order) are characterized by **essential implications** (the simplest consequences of the mixed transitivities), by the properties of the functions in the **value-threshold support**, by the circuits in the **graph representation** and by the shapes of the **matrix representation**. The mixed transitivity is the most natural way of introducing the structures of the systems. The value-threshold and the graph and matrix representations are also useful for understanding them. In addition, the tools designed for the transitivity closure, for paths in graphs, and for linear constraints on matrix entries are adaptable to implement and manage efficiently the preference relational systems in **Computer Aided Decision Support Systems** by using simple software.

1 Introduction.

Every decision must be based on criteria that show the **preferences** of the decision system on the set of alternatives. **Preference systems** are models used to formalize and manage the preference information. **Preference Relational Systems** are preference systems based on a series of exhaustive and mutually exclusive binary relations. **Binary relations** are the most suitable mathematical tool to modelize preferences based on paired comparisons between alternatives. They are represented by graphs and boolean matrices that give a clear vision of the structure of the relation and are easily manipulated by computers using simple software. Next section reviews the main ideas concerning binary relations.

A general preference relational systems consists of four binary relations: the **preference** P, the **semipreference** S, the **indifference** I and the **incomparability** X. The preference and the indifference appear in classic theory and the semipreference can be understood by a degree or precision question between the preference and the indifference. The incomparability corresponds to the nonexistence of enough information to relate the alternatives. The preference system is **complete** when the incomparability is excluded. The preference relational system is **simple** when every comparable pair of alternatives are indifferent or one of them is preferable. So the classical model is the complete simple preference relational system. These preference relational systems are defined in section 3.

The **transitivity** plays a very important role in the modelization of preferences with binary relations. In the classic theory, the transitive property is usually imposed to the preference relation and sometimes to the indifference relation. The transitivity of the indifference relation is frequently replied with realistic paradoxes of indifference chains, where the first and last alternative are clearly not indifferent. However, it looks natural that, given a chain of preference and indifference relations, where the preference relations are significantly more important than the indifference ones, the first and last alternatives of the chain are in a preference situation. The degree or intensity of the preference between the pair of alternatives joined by the chain, must be a function of the intensity, weight and number of relations that compose the chain. The conditions to conclude a preference relation must guarantee the avoiding of the possibility of chains with extremes that are close to indifference circumstances.

These reasons motivate several conditions, similar to the transitivity but including different types of relations, in order to avoid cycles of relations that indicate a clear contradiction. They are known as **Mixed Transitivity** properties and are formulated by conditions imposed to a chain of preference and indifference relations to conclude the preference between the first and the last alternative. These conditions are expressed on the number, kind and position of the preference relations within the chain. Therefore **mixed transitivities** are used to define the structures of a preference relational system in section 4. A mixed transitivity property is the set of conditions under which a chain of semipreference, preference and/or indifference relations implies the preference (or semipreference) between the first and the last alternative in the chain. The most important structures of a preference relational system : three for simple systems (the **preorder**, the **semiorder** and the **interval order**) and three for general systems (the **quasiorder**, the **double semiorder** and the **pseudo-order**) are here considered (see Roy (1985), Roubens and Vincke (1985), Roy and Vincke (1987), Moreno (1992)).

The mixed transitivities are useful to know the whole preference system without comparing every pair of alternatives. It is also important to know the sets of paired comparisons that are sufficient to infer the whole preference relational system. Each of these six structures is characterized by a set of **essential implications** that are the corresponding mixed transitivities applied to the shortest chains. They constitute the shortest definition of the structure that is convenient from mathematical point of view. The sets of essential implications corresponding to each of the structure are given in section 5 (the proofs of the propositions are outlined in the Aœppendix).

A usual way of adopting a preference system is by using value and threshold functions on the set of alternatives. A **value-threshold support** of a preference relational system consists of three functions on the set of alternatives that characterize the system: a **value** function g, a **preference threshold** function p and an **indifference threshold** function q. The support determine the relation between two alternatives a and b by: *(i)* a is preferable to b when the value of a is greater than the preference threshold of b; *(ii)* a is indifferent to b when the value of a is not greater than the indifference threshold of b and *(iii)* a is semipreferable to b when the value of a is greater than the indifference threshold of b and not greater than its preference threshold. The incomparability appears when some of the corresponding functions are not evaluated. The six structures of preference relational system are directly related with the properties of the possible value-threshold supports. Section 6 give the characterization of the structures by the properties of these supports.

Preference relational systems can be represented by graphs and by matrices. They are very useful for obtaining a clear vision of the preference system. The graph representation consists in a multi-linked graph with several types of arcs and edges. Each of the structures of a complete preference relational system is characterized by the nonexistence of some circuits in its graph representation. The matrix representation consists of a set of boolean matrices or of a multi-valued matrix. The structure of the complete preference system is also characterized by the shapes of the possible matrix representation with appropriate choices for the assignment of the alternatives to the rows and to the columns. Sections 7 and 8 give the characterization of complete preference relational systems by the properties of the possible graph and matrix representation.

In the conclusions, the key ideas of the advantages, management tools and efficient implementation of preference relational systems are shown. These are the main criteria for using them in a **Computer Aided Decision Support System**. The two sides of the advantages of these systems are: the intuitive way of introducing them and the possibility of efficient implementations to manage them. The two main types of management tools consists of operations to obtain the relations implicit in each structure and the constraints corresponding to these structures.

Every essential implication gives an operation on the preference relational system similar to the composition for the transitivity. In a similar way of the transitivity closure, given a set of compared pairs, the smallest preference system with each of this structure is obtained by iteratively applying the operations corresponding to its essential implications. The structures of the preference systems are also given by a set linear constraints on the entries of the matrices associated with the system. This is very useful to use mathematical programming tools to manage the preference relational systems. The efficient implementation must be obtained taking into account these sets of operations and linear constraints.

Finally, the proofs of the propositions included in the paper are outlined in the appendix.

2 Binary Relations.

Let A be a set of **Alternatives** in its most generic meaning. The elements of this set can be any material objects, quantities of money or of any other economic good, candidates, noises, social conditions, illnesses, time, resources. A **Binary Relation** R on set A describes the existence of a connection property between pairs of alternatives. For any alternatives $a, b \in A$, aRb means that alternative a is related by R with alternative b. Then the main properties of binary relations are:

Reflexitivity:	$aRa,$	$\forall\, a \in A.$
Irrefletixivity:	$\neg aRa,$	$\forall\, a \in A.$
Symmetry:	$aRb \Rightarrow bRa,$	$\forall\, a, b \in A.$
Antisymmetry:	$aRb \Rightarrow \neg bRa,$	$\forall\, a, b \in A.$
Linearity:	$\neg aRb \Rightarrow bRa,$	$\forall\, a, b \in A.$
Transitivity:	$aRbRc \Rightarrow aRc,$	$\forall\, a, b, c \in A.$

The two most important types of binary relations are the **Orders** (reflexive, antisymmetric and transitive relations) and the **Equivalences** (reflexive, symmetric and transitive relations). Some useful operations with binary relations on a set A are the following:

The **composition** defined by:	$aRSb$	if and only if	$aRcSb$ for some $c \in A$.
The **union** defined by:	$a(R \cup S)b$	if and only if	aRb or aSb.
The **intersection** defined by:	$a(R \cap S)b$	if and only if	aRb and aSb.
The **converse** defined by:	$aR^{-1}b$	if and only if	bRa.
The **complementary** defined by:	$a\overline{R}b$	if and only if	$\neg aRb$.

3 General Preference Relational Systems.

A **Decision System** uses the available information to establish the preferences on the alternatives. For every pair of alternatives the decision system shows its preferences by choosing among a series of possible situations: one of the alternatives is preferable, is almost preferable, is semipreferable, they are indifferent, are almost indifferent, etc. (see Roy, 1985). These situations motivate some binary relations on the alternative set. The resulting binary relations constitute a **Preference Relational System**. The structure of the preference relational system is given by the properties of these set of binary relations.

Some preference relational systems are obtained by modifying other preference relational systems in two directions: in one hand, by relaxing the constraints of the binary relations, by introducing intermediate relations or by decomposing or grouping two (or more) relations; in the other hand, by excluding or eliminating ambiguous relations or by adding some constraints to the set of binary relations. The purpose of these modifications is to allow a bigger slack in the specification of the situation corresponding to each pair of alternatives or to avoid ambiguous or conflictive consequences.

The classical theory allows for any two alternatives, the possibility of choosing between the **Preference** of one of the alternatives and the **Indifference** between them. This model is extended by allowing one to choose a **Semipreference** when there is doubt between these two possibilities, for instance because there are some reasons to prefer one of them that are not enough significant. Furthermore, it is possible that, for a given pair of alternative, there is no information about the preference; therefore there is an **Incomparability** relation between them. So the new model considers four binary relations on the alternative set: the preference P, the semipreference S, the indifference I and the incomparability X. Clearly, P and S must be antisymmetric and I and X are symmetric; I is reflexive but X is irreflexive. Thus, given two alternatives a and b, it is possible to select one and only one of the six possibilities:

aPb, aSb, $aIb(= bIa)$, bSa, bPa and $aXb(= bXa)$.

Definition 1 *A* **General Preference Relational System** *on alternative set A is a preference relational system consisting of a 4-tuple (P, S, I, X) of exhaustive and mutually exclusive binary relations on A such that I is reflexive, I and X are symmetric, and P and S are antisymmetric.*

However in an ideal context the two new binary relations, the semipreference and the incomparability, would be excluded.

Definition 2 *A preference relational system is* **Simple** *if the semipreference is excluded. A preference relational system is* **Complete** *if the incomparability is excluded.*

Therefore, a **Simple Preference Relational System** consists of a 3-tuple (P, I, X) of exhaustive and mutually exclusive binary relations such that I is reflexive, I and X are symmetric, and P is antisymmetric. A **Complete Preference Relational System** consists of a 3-tuple (P, S, I) of exhaustive and mutually exclusive binary relations such that I is reflexive and symmetric, and P and S are antisymmetric. Furthermore, a **Complete Simple Preference Relational System** consists of a pair (P, I) of exhaustive and mutually exclusive binary relations such that I is reflexive and symmetric, and P is antisymmetric.

Every simple preference relational system is determined by its **Characteristic** relation C, defined by $C = P \cup I$. If C is the characteristic relation of (P, I, X) then: $I = C \cap C^{-1}$, $P = C \cap \overline{C}^{-1}$ and

$X = \overline{C} \cap \overline{C}^{-1}$. A simple preference relational system is complete if the characteristic relation C is a linear relation. In a complete simple preference relational system, the preference relation P also characterizes the system. The complete general preference relational system is not fully characterized by only one binary relation. A general preference relational system is complete if and only if $P \cup S \cup I$ is a linear relation.

Every general preference relational system has two associated simple preference relational systems that are respectively obtained by joining the semipreference to the preference and to the indifference. The **Extended Preference** relation is $T = P \cup S$ which produces the first simple preference relational system with characteristic relation $D = T \cup I = P \cup S \cup I$. The **Extended Indifference** relation is $J = S \cup I \cup S^{-1}$ that produces the second simple preference relational system with characteristic relation $E = P \cup J = P \cup S \cup I \cup S^{-1}$. The simple preference relational systems associated with a general preference relational system (P, S, I, X) are the two systems (T, I, X) and (P, J, X).

In several applications, some relations appear from the union of other simpler relations. Therefore, choosing one of these situations, versus choosing one of the simpler situations that compound it, is a precision question in the judgment but they are not contradictory situations. For instance, the assertions aPb and aTb are different but both agree that aTb. This is very important in applying preference relational systems in computer-aided group and multicriteria decision making.

4 Structures of Preference Relational Systems.

The structure of the preference relational system is given by the number of binary relations and the properties or constraints verified by them. Three of the most important structures of a simple preference relational system (P, I, X), the **preorder**, the **semiorder** and the **interval order**, are defined by using mixed transitivity properties. A mixed transitivity property in a simple preference relational system (P, I, X) is given by the set conditions to conclude aPb from a chain like $aCxCyC...CzCb$. This chain is denoted in short as $aC....Cb$.

Definition 3 *The structure of the simple preference relational system (P, I, X) is:*

1. *The* **Preorder** *structure if and only if:*
   If $aC....Cb$, with (at least) a P, then aPb.
2. *The* **Semiorder** *structure if and only if:*
   If $aC....Cb$, with more P than I, then aPb.
3. *The* **Interval Order** *structure if and only if:*
   If $aPC....CPb$, without two consecutive I, then aPb.

Also, three of the most important structures of a general preference relational system, the **quasi-order**, the **double semiorder** and the **pseudo-order** are also defined by mixed transitivity properties. A mixed transitivity property in a general preference relational system (P, S, I, X) is a set of conditions to conclude aPb or aTb from a chain like $aDxDyD...DzDb$ or like $aExEyE...EzEb$. Each of these structures is defined with three mixed transitivities, one corresponding to the first simple preference relational system (T, I, X), another corresponding to the second simple relational system (P, J, X) and the last one to the whole general system (P, S, I, X).

Definition 4 *The structure of the general preference relational system (P, S, I, X) is:*

1. *The* **Quasi-order** *structure if and only if:*
   (a) *If $aD....Db$, with at least one T, then aTb.*
   (b) *If $aE....Eb$, with more P than J, then aPb.*
   (c) *If $aD....Db$, with at least one P, then aPb.*
2. *The* **Double Semiorder** *structure if and only if:*
   (a) *If $aD....Db$, with more T than I, then aTb.*

(b) If $aE....Eb$, with more P than J, then aPb.
(c) If $aD....Db$, with at least one P and more T than I, then aPb.
3. *The* **Pseudo-order** *structure if and only if:*
(a) If $aD....Db$, with more T than I, then aTb.
(b) If $aE....Eb$, with more P than J, then aPb.
(c) If $aD....DPb$ or $aPD....Db$, without two consecutive J, then aPb.

Note that every preorder is also a semiorder and every semiorder is an interval order. Every quasi-order is a double semiorder, every double semiorder is a pseudo-order. The two simple preference relational systems associated with a quasiorder are a preorder and a semiorder, those associated with a double semiorder are semiorders and those associated with a pseudo-order are also two semiorders

5 Essential Implications.

It is very important to know the simplest allowed chains in every mixed transitivity property. The **Essential Implications** of each structure is a set of implications (obtained from the simplest allowed chains in every mixed transitivity property) that characterizes the corresponding structure. The preorder has three essential implications with length 2, the semiorder has three essential implications with length 3 and the interval order has only one essential implication with length 3.

Proposition 1 *The essential implications in a simple preference relational system (P, I, X) are:*
1. *In the preorder structure:*
$aPbPc \Rightarrow aPc$, $aPbIc \Rightarrow aPc$, $aIbPc \Rightarrow aPc$.
2. *In the semiorder structure:*
$aPbIcPd \Rightarrow aPd$, $aPbPcId \Rightarrow aPd$, $aIbPcPd \Rightarrow aPd$.
3. *In the interval order structure:*
$aPbIcPd \Rightarrow aPd$.

The proofs of this proposition and all the propositions included in this paper are out of the scope of this paper, then they are outlined in the appendix.

Note that in a preorder, relation I could not be transitive. In a semiorder, it could be $aPIb$ or $aIPb$ but not aPb. And, in an interval order, it could be $aPPIb$ or $aIPPb$ but not aPb.

Each of the structures of a general preference relational system has also a set of essential implications that characterizes it. Some of these implications are ambiguous because they conclude not with a binary relation of the preference relational system but with the union of some of them. They are not ambiguous when applied in one of the two associated simple preference relational systems. The essential implications of the structures of general systems are grouped in three sets: those of *(a)* correspond to the simple preference relational system (T, I, X), those of *(b)* correspond to the simple system (P, J, X) and those of *(c)* to the general system (P, S, I, X). The quasi-order has seven essential implications with length 2 and three implications with length 3, the double semiorder has twelve essential implications with length 3 and the pseudo-order has ten implications with length 3.

Proposition 2 *The essential implications in a general preference relational system (P, S, I, X) are:*
1. *In the quasi-order structure:*
(a) $aTTb \Rightarrow aTb$, $aTIb \Rightarrow aTb$, $aITb \Rightarrow aTb$.
(b) $aPJPb \Rightarrow aPb$, $aJPPb \Rightarrow aPb$, $aPPJb \Rightarrow aPb$.
(c) $aPIb \Rightarrow aPb$, $aIPb \Rightarrow aPb$, $aPSb \Rightarrow aPb$, $aSPb \Rightarrow aPb$.
2. *In the double semiorder structure:*
(a) $aTITb \Rightarrow aTb$, $aITTb \Rightarrow aTb$, $aTTIb \Rightarrow aTb$.

(b) $aPJPb \Rightarrow aPb$, $aJPPb \Rightarrow aPb$, $aPPJb \Rightarrow aPb$.
(c) $aPSIb \Rightarrow aPb$, $aSIPb \Rightarrow aPb$, $aISPb \Rightarrow aPb$,
$aPISb \Rightarrow aPb$, $aIPSb \Rightarrow aPb$, $aSPIb \Rightarrow aPb$.

3. *In the pseudo-order structure:*
(a) $aTITb \Rightarrow aTb$, $aITTb \Rightarrow aTb$, $aTTIb \Rightarrow aTb$.
(b) $aPJPb \Rightarrow aPb$, $aJPPb \Rightarrow aPb$, $aPPJb \Rightarrow aPb$.
(c) $aPSIb \Rightarrow aPb$, $aSIPb \Rightarrow aPb$, $aPISb \Rightarrow aPb$, $aISPb \Rightarrow aPb$.

In a quasi-order, relations T, P and I are transitive but neither S nor J has to be transitive. In a double semiorder it could be $aIPIb$ but not aPb. And, in a pseudo-order it could be $aIPSb$ or $aSPIb$ but not aPb.

6 Value-Threshold Support.

A usual way of adopting a preference system is by using value and threshold functions on the set of alternatives. A **value-threshold support** of a preference relational system consists of three functions on the set of alternatives that characterize the system: a **value** function g, a **preference threshold** function p and an **indifference threshold** function q.

Definition 5 *A* **Value-Threshold support** *of a preference relational system is a 3-tuple (g, p, q) of functions on the alternatives such that for every $a, b \in A$:*

$$\begin{aligned} aPb &\Leftrightarrow p(b) < g(a). \\ aSb &\Leftrightarrow q(b) < g(a) \leq p(b). \\ aIb &\Leftrightarrow g(a) \leq q(b) \text{ and } g(b) \leq q(a). \end{aligned}$$

Every value-threshold support verifies: $g(a) \leq q(a) \leq p(a),\ \forall\ a \in A$. The incomparability appears when some of the corresponding functions are not yet evaluated; the supported preference relational systems are complete. The properties of the value-threshold support determine the structure of the preference relational system.

Proposition 3 *The structure of a supported general preference relational (P, S, I, X) system is:*
1. *The quasi-order structure if:*
The difference between the preference threshold and the value can be chosen constant and the indifference threshold equals the value.
2. *The double semiorder structure if:*
Both differences between the thresholds and the value can be chosen constant.
3. *The pseudo-order structure if:*
One of the differences between each threshold and the value can be chosen constant.

If both thresholds are equal then the preference relational system is simple. Also the properties of the value and the indifference-preference threshold of the support determine the structure of a simple preference relational system.

Proposition 4 *The structure of a supported simple preference relational system is:*
1. *The preorder structure if the threshold equals the value.*
2. *The semiorder structure if the difference between the threshold and the value is constant.*
3. *The interval order structure if the threshold is a function of the value.*

Moreover, if the set of alternatives is finite then given a preference relational system with one of these structures, there exist a support verifying the corresponding conditions according to the last two propositions.

7 The Graph Representation.

The **Associated Graph** $G(R)$ of a binary relation R is obtained by representing the alternatives of set A as vertices or nodes and by joining with arcs the alternatives related by R; arc (a, b) is in directed graph $G(R)$ if aRb. A symmetric binary relation can be represented by an undirected graph by using edges instead of arcs. Therefore, a preference relational system is represented by using several kinds of edges or arcs corresponding to the binary relations of the system. Since the binary relations are exhaustive and mutually exclusive, a preference relational system is represented by a complete mixed or multi-linked graph. The graph representation of a general preference relational system has P-arcs, S-arcs, I-edges and X-edges. Furthermore, one can talk of T-arcs, J-edges, C-circuits, D-circuits, E-circuits and even S^{-1}-arcs. Also, the properties of the graph representation give the structure of the complete systems.

Proposition 5 *The graph representation of a complete simple preference relational system (P, I) has the following properties:*

1. *For the preorder structure:*
   - There are not C-circuits with at least one P-arc.
2. *For the semiorder structure:*
   - There are not C-circuits with at least a half of P-arcs.
3. *For the interval order structure:*
   - There are not C-circuits without at least two consecutive I-edges.

Proposition 6 *The graph representation of a general preference relational system (P, S, I) has the following properties:*

1. *For the quasi-order structure:*
   (a) There are no D-circuits with at least one T-arc,
   (b) There are no E-circuits with at least a half of P-arcs, and
   (c) There are no E-circuits with at most one S^{-1}-arc and at least one P-arc.
2. *For the double semiorder structure:*
   (a) There are no D-circuits with at least a half of T-arcs,
   (b) There are no E-circuits with at least a half of P-arcs and
   (c) There are no E-circuits with
   at most one S^{-1}-arc, more T-arcs than I-edges and at least one P-arc.
3. *For the pseudo order structure:*
   (a) There are no D-circuits with at least a half of T-arcs,
   (b) There are no E-circuits with at least a half of P-arcs and
   (c) There are no E-circuits with
   at most one S^{-1}-arc, at least a P-arc in each two consecutive arcs.

These graph properties also characterize the six structure for complete systems; that is, the complete preference relational system has one of these structures if the graph representation verifies the corresponding properties according to the last two propositions.

8 The Matrix Representation.

Let A be a set of n alternatives. The **Associated Matrix** $M(R)$ of a binary relation R is a squared boolean or 0-1 matrix of size n that represents the relation. Every alternative $a \in A$ is assigned to row $r(a)$ and to column $c(a)$. The entry of $M(R)$ in row $r(a)$ and column $c(b)$ is 1 if aRb and it is 0 otherwise. The shape of the associated matrix of a binary relation depends on its properties and these functions $r(.)$ and $c(.)$. The value 1 can be substituted by another symbol; for instance by the name R of the binary relations. Then a preference relational system is represented by the corresponding set of associated matrices or by a mixed or multi valued matrix. The structure of the preference relational system is also determined by the possible shapes of the matrix representation by choosing appropriate row and column assignment.

A boolean or 0-1 matrix $M = [m_{ij}]$ is **Full Triangular** when $m_{ij} = 0$ for $j \leq i$ and $m_{ij} = 1$ for $j > i$. The matrix associated with an order is full triangular using row and column assignment according to the order. A matrix M is **Full Upper-Triangular** if there is a nondecreasing integer function $k(.)$ (with $k(i) \geq i$) such that $m_{ij} = 0$ for $j \leq k(i)$ and $m_{ij} = 1$ for $j > k(i)$. The adjective **"box"** is used when the members of a partition of the rows (and the columns) are in the role of the rows (and the columns) in these definitions. For example, with appropriate row and column assignments, the associated matrix of an equivalence is box diagonal.

Proposition 7 *The matrix representation of a complete simple preference relational system (P, I) has the following possible shapes:*

1. *For the preorder structure:*
   $M(P)$ is full box triangular, with one alternative assignment $r = c$.
2. *For the semiorder structure:*
   $M(P)$ is full upper-triangular, with one alternative assignment $r = c$.
3. *For the interval order structure:*
   $M(P)$ is full upper-triangular, with two alternative assignments r and c.

For the representation of a general preference relational system the possible shapes of $M(P)$ and $M(T)$ give the structure of the system. Let r_P and c_P denote the row and column assignments for $M(P)$ and let r_T and c_T denote the assignments for $M(T)$.

Proposition 8 *The matrix representation of a complete general preference relational system (P, S, I) has the following properties:*

1. *For the quasi-order structure:*
   $M(T)$ is a box triangular matrix and $M(P)$ is a full upper-triangular matrix,
   with only one alternative assignment $r_T = r_P = c_T = c_P$.
2. *For the double semiorder structure:*
   $M(T)$ and $M(P)$ are two full upper-triangular matrices,
   with one alternative assignment $r_T = r_P = c_T = c_P$.
3. *For the pseudo order structure:*
   $M(T)$ and $M(P)$ are two full upper-triangular matrices,
   with two alternative assignments $r_T = r_P = c_T$ and c_P (or $r_T = r_P = c_P$ and c_T).

9 Conclusions.

9.1 Advantages of Preference Relational Systems.

The preference relational system is one of the possibilities for representing and managing the preferences of a decision system. The main advantages of the preference relational systems are derived from the mathematical basis of the binary relations.

Binary relations are perfectly appropriate to formalize the preferences based on paired comparisons. It is a mathematical basic concept that has been extensively studied from several points of view. In addition, binary relations are usually represented and implemented by graphs and matrices, that can be easily manipulated by computers using simple software. The preference relational system is a model that also has these two advantages of binary relations: their natural interpretation and the wide set of adaptable software.

The paired comparisons are the most obvious way to show the preference feeling of a decision maker or an expert. These are also usually the most direct data obtained from comparative analysis in practical research (for example, see David, 1988). The transitivity is also a very intuitive idea that is very easy to understand. According to the frequent objections to the transitivity of the indifference, mixed transitivities allows several degrees in the relaxation of the transitivity. So the mixed transitivity, that can be understood as an extension of the transitivity of binary relations, is a very good way of introducing the structures of preference relational systems to everyone. The essential implications show to the users the best way to test the structure of a preference relational system. The value-threshold supports constitute a instinctive manner of elucidating the preferences. The six structures here considered are directly related with simple value-threshold support properties. Graphs and matrices are rudimentary tools to show the information that are familiar in every application context.

Binary relations, transitivity, implications, functions, graphs and matrices are elementary mathematical concepts. Their properties have been studied from a large number of points of view. Transitivity is one of the property of binary relations most extensively studied. Implications and functions are widely used in most fields of Mathematics. The essential implications constitute the best characterizations of the structures from the mathematical point of view because they are the shortest and most precise conditions. Graphs and matrices are usual graphical arrangements of information. Furthermore, the tools designed for transitivity of binary relations, paths and circuits in graphs and efficient operation and storage of matrices can be easily adapted to preference relational systems.

Use of representations of the preference relational systems is very important for getting a clear global view of their structure and simplifying the analysis. A binary relation is represented by a directed graph or by an undirected graph if it is symmetric. Therefore, a preference relational system is represented by a mixed or multi-linked graph with arcs and edges. Thus, a simple preference relational system is represented by a mixed graph in which the arcs represent the preference relation P, the edges represent the indifference relation I and the incomparability is represented by the nonexistence of links between the corresponding alternatives. The graph representation of a general preference relational system needs two kinds of arcs, one for the preference P and another one for the semipreference S. Although, an edge or arc could be omitted when it is fixed by hypotheses (like indifference loops) or when it obviously follows from the rest of them in the structure (by mixed transitivity) to improve the clarity.

9.2 Management Tools.

The transitivity allows the inferring of relations between alternatives; aPc can be concluded from $aPbPc$. This is practical because it is not necessary to have all pairs compared in the research and in the representation. The transitivity give an operation to obtain these conclusions that is performed by adding to P the composition PP, i.e. $P \leftarrow P \cup PP$. This operation can be implemented by:

    for $a, b, c \in A$ **do**:
        if aPb and bPc **then** aPc.

It can be iteratively applied to any binary relation until one obtains the **transitivity closure** of the binary relation; i.e., the smallest transitive relation that includes it.

    repeat
        $P \leftarrow P \cup PP$
    until (P is not changed).

See Sedgewick (1983) for algorithms and applications of the transitivity closure.

A mixed transitivity property also can be understood as an implication that allow the inferring of relations between alternatives. For instance, with the mixed transitivity of interval orders, aPd can be concluded from $aPbIcPd$. Thus, the performing of some paired comparisons can be skipped when the result can be inferred from the properties and previous comparisons. Each of the essential implications give an operation that can be applied iteratively, the relations obtained are implicit in the chosen structure, like with transitivity property. For instance, the operation corresponding to the interval order mixed transitivity is: $P \leftarrow P \cup PIP$. This operation can be implemented by:

for $a, b, c, d \in A$ **do**:
  if aPb, bIc and cPd **then** aPd.

The mixed transitivity closure corresponding to these six structures are defined similarly to transitivity closure. Thus, the interval order closure of a set of paired comparisons is the smallest interval order that includes the outcomes of the comparisons. It is obtained by the following algorithm:

repeat
  $P \leftarrow P \cup PIP$
until (P is not changed).

Since each of these six structures is characterized by essential implications, the corresponding mixed transitivity closure of a preference relational system is defined in a similar way.

The mixed transitivity properties can be interpreted on the associated graph concerning paths. Besides, the structure of the complete systems is characterized by the (non)existence of certain circuits. So the tools for searching for paths or circuits in a graph can be adapted to show the properties of binary relations and the structure of preference relational systems.

A binary relation can also be represented by a boolean matrix by assigning the columns and the rows to the alternatives. These assignment are useful mainly to show graphically the structure of the system through the matrix shape. The properties of the binary relation can also be characterized as linear constraints on the entries of the matrix, whatever assignment used. Let $[p_{ab}]_{a,b\in A}$ denote the boolean matrix associated with relation P; i.e. aPb if and only if $p_{ab} = 1$. Analogously the associated matrices of all the binary relations in the preference relational systems are $[i_{ab}]$, $[s_{ab}]$, $[t_{ab}]$, etc. Then the transitivity of a binary relation is characterized by a set of linear constraints on its associated matrix $[m_{ab}]$ as follow:

$$m_{ab} + m_{bc} - m_{ac} \leq 1, \quad \forall\, a, b, c \in A.$$

Analogously, the set of linear constraints corresponding to the mixed transitivity of interval order is:

$$p_{ab} + i_{bc} + p_{cd} - 2p_{ad} \leq 1, \quad \forall\, a, b, c, d \in A.$$

Similar set of constraints are obtained for every essential implication of the six structure.

By using these constraints, very useful characterizations of the main types of binary relations are obtained. A order on A is a binary relation of which the associated boolean matrix verifies:

$$\begin{aligned} m_{aa} &= 1, \quad \forall\, a \in A. \\ m_{ab} + m_{ba} &= 1, \quad \forall\, a, b \in A. \\ m_{ab} + m_{bc} - m_{ac} &\leq 1, \quad \forall\, a, b, c \in A. \end{aligned}$$

In the same way, the matrix representation of a simple preference relational system (P, I, X) verifies the constraints:

$$\begin{aligned} i_{aa} &= 1, \quad \forall\, a \in A. \\ i_{ab} - i_{ab} &= 1, \quad \forall\, a, b \in A. \\ p_{ab} + p_{ba} + i_{ab} &\leq 1, \quad \forall\, a, b \in A. \end{aligned}$$

The system is complete if

$$p_{ab} + p_{ba} + i_{ab} \geq 1, \quad \forall\, a, b \in A.$$

And then, the system has the interval order structure if

$$p_{ab} + i_{bc} + p_{cd} - 2p_{ad} \leq 1, \quad \forall\, a, b, c, d \in A.$$

Therefore the characterization of the associated matrices of a complete interval order by linear constraints is:

$$\begin{aligned} i_{aa} &= 1, \quad \forall\, a \in A. \\ i_{ab} &= i_{ba}, \quad \forall\, a, b \in A. \\ p_{ab} + p_{ba} + i_{ab} &= 1, \quad \forall\, a, b \in A. \\ p_{ab} + i_{bc} + p_{cd} - 2p_{ad} &\leq 1, \quad \forall\, a, b, c, d \in A. \end{aligned}$$

In the same way, every essential implication has a corresponding class of linear constraints. The set of linear constraints corresponding to the essential implications that characterize each structure also characterizes the matrix representation of such structure. Therefore each of the structure of the preference relational systems is characterized by a set of linear constraints.

9.3 Efficient Implementation

Let n be the number of alternatives. To store a complete preference relational system, $O(n^2)$ memory is needed and the value-threshold support takes only $O(n)$ memory. So if it is possible to use a support then the management of the preference system is more efficient.

The number of constraints corresponding to every essential implication is the number of alternative up to one plus the length of the essential implication. Therefore the number of linear constraints that characterized each of the structure is obtained from the lengths of the essential implications. The number constraints corresponding to the complete simple preference relational system is $O(n^2)$. For instance, the number of constraints for the interval order structure is n^4 and the double semiorder structure is characterized by $12n^4$ linear constraints. The number of linear constraints that characterizes any of the structures considered in this paper is $O(n^4)$ but the preorder is $O(n^3)$. These characterizations are very useful to use mathematical programming tools to manage the corresponding type of binary relation and the preference relational system.

10 Appendix.

This appendix outlines the proofs of the propositions included in the paper.

Proposition 1 and 2 states the necessity and sufficiency of the essential implications of each of the six structures. These implications are special cases of mixed transitivity, then they are obviously true, therefore they are necessary conditions. In the other hand, given the chain verifying one of the mixed transitivity conditions of the structures, there is one of these implications that reduces the length of the chain. Then they are also sufficient conditions.

Propositions 3 and 4 states the structure of the preference relational system given by the value-threshold support properties. This is shown by proving the essential implications of the corresponding structure. Furthermore, if the set A of alternatives is finite then, given a preference relational system with one of the six structures, there exists a support with the corresponding properties according to propositions 3 and 4.

The existence of a value-threshold support for a preorder is a classic result; the value of any alternative a is the number of alternatives to which a preferable. Scott and Suppes (1958) proved the semiorder support that is obtained using shortest path length on graph $G(T)$ (see, Roubens y Vincke (1985)). Fishburn (1970) gave the interval order support; the value of an alternative a is obtained by using the number of alternatives to which a preferable and the number of alternatives that are preferable to a.

The value-threshold support of a quasi-order is obtained by considering the resulting simple preference relational system in the quotient set A/I (since I is an equivalence relation) that is a semiorder. The support of the double semiorder is given by Cozzens and Roberts (1982) using a definition that is equivalent to the definition used here (see Moreno (1992)). The two supports of a pseudo-order are given by Vincke (1980) and by Roy and Vincke (1987).

The characterization by the properties of the graph representations given in proposition 5 and 6 are obtained by interpreting the corresponding mixed transitivities as the nonexistence of circuits. These circuits are obtained by joining the alternatives in the extremes of the mixed transitivity chain with the negation of the corresponding inferred relation (since the system is complete). For instance, in a complete interval order, $aPIPb \Rightarrow aPb$ is equivalent to the nonexistence of a circuit $aPIPbCa$. Therefore, the nonexistence of a C-circuit without two consecutive I- arcs is a necessary and sufficient condition.

The shapes of matrix representations given in propositions 7 and 8 are obtained by choosing appropriate assignments. The row assignment of the alternatives is made always with non-decreasing value function and the column assignment with non-decreasing threshold function. The box in the systems that include a preorder (preorder, quasi-order and interval preorder) are given by the equivalence classes of relation I. The preference threshold function is used in the column assignment for the shape of $M(P)$ and the indifference threshold function for the matrix associated with T when needed.

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