Relaxation of the Condorcet and Simpson Conditions in Voting Location*

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Abstract

A Condorcet point, in voting location, is a location point such that there is no other closer to more than half of the users. However, such Condorcet solution does not necessarily exist. This concept is based on two assumptions. First, two locations are indifferent only if they are at the same distance of the voter. Second, the number of voters needed to reject a location is more than half of them. We relax the Condorcet condition in two ways. First, by considering that two locations are indifferent for every user if the difference of the distances to them is within a positive threshold. Secondly, by considering that the proportion of users needed to reject a location is not one half. We consider the resulting new solution concepts that arise by applying both relaxations at the same time and develop algorithms for obtaining them in the finite case.

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1 Introduction.

Some optimal decisions about the location of facilities can be considered as the result of a voting process among the users. For desirable facilities, each user wishes to have the facility as close as possible. Several outranking procedures or group decision rules have been considered for selecting a location point taking into account the preferences of a set of voters [?], [?]. One of the most usual rules is the Condorcet rule that consists of selecting the location such that no other location is preferred to it by a strict majority of voters. Therefore, for a desirable facility, a location is a Condorcet solution if there is not another location closer to more than a half of the set of users [?].

However, a Condorcet location does not necessarily exist; i.e., there are circumstances where the set of Condorcet points is empty. For instance, the Condorcet set is empty in a network consisting of an equilateral triangle with one user at each vertex. Then the Condorcet conditions can be relaxed in two ways: by increasing the proportion of users that have to prefer another location in order to reject a point as a possible solution, or by increasing the threshold of the distance for considering a location preferred to another one for every user. The former possibility provides the Simpson location [?] and the latter provides the Tolerant Condorcet location [?]. A review for the network case can be found in [?] and for the continuous case in [?]. Other interesting references on this field are [?], [?], [?], [?], [?] and [?].

We introduce two parameters in the model: a tolerance distance α and a qualified majority γ. The tolerance distance α is a threshold for the indifference of every user, providing a preference structure named semiorder [?]. Then two locations x and y for the facility are indifferent for a voter if the distances to x and to y differ in at most α. The qualified majority γ is the proportion of users preferring another location above which a point is rejected. The $[\alpha,\gamma]$-Condorcet locations are those not rejected using these two parameters in the rejection rule.

In the rest of the section we formulate the corresponding family of voting problems. Next section shows the proposed solution procedures for the finite case and provides polynomial algorithms to solve the problems. And we finish with a section that illustrates the results by an example and a conclusion section.

Consider that a facility point has to be selected in a set of possible locations L to serve a set of users U that want the facility as close as possible.

Let $d(u, x)$ denote the distance from the user $u \in U$ to the location point $x \in L$. The distances to the users can be interpreted as a family of normalized criteria or value functions for a general decision problem. If the location is selected avoiding the possibility of being rejected by a majority of users, then the Condorcet location arises as follows.

Definition 1 *A* **Condorcet location** *is a location such that no other location is closer to more than a half of the users.*

Let C denote the set of Condorcet locations. Then

$$C = \{x \in L : |\{u \in U : d(y, u) < d(x, u)\}| \leq |U|/2, \forall y \in L\}.$$

This solution concept considers, on the one hand, that two locations x and y are indifferent for every user if and only if they are at the same distance from the user, and on the other hand, that the majority of users that are needed to reject a location is more than half of them. Since, for a large set of instances there is no Condorcet solution, the constraints seem to be very hard and they could be relaxed.

Then consider that the strict majority of voters that have to be against a solution to be rejected is more than a proportion γ instead of $1/2$. Also consider that, for every user, two locations are indifferent if the difference of their distances to the user is within a given threshold α. Then every user u prefers y to x if the distance to x is bigger than α plus the distance to y; i.e., $d(u, y) + \alpha < d(u, x)$. By considering the qualified majority γ and the tolerance distance α we obtain a family of solution concepts named the $[\alpha,\gamma]$-Condorcet locations.

Definition 2 *A location x is an $[\alpha,\gamma]$-***Condorcet location** *if and only if, for every other location y, the proportion of users closer to y than to x by more than a distance α, is not greater than γ.*

Let $C(\alpha, \gamma)$ denote the set of $[\alpha,\gamma]$-Condorcet locations with tolerance distance α and qualified majority γ. Then

$$C(\alpha, \gamma) = \{x \in L : |\{u \in U : d(y, u) < d(x, u) - \alpha\}| \leq \gamma |U|, \forall y \in L\}.$$

Proposition 3 *The family of sets $C(\alpha, \gamma)$ is non decreasing with respect to both parameters.*

Proof. First we show that $\alpha_1 < \alpha_2 \Rightarrow C(\alpha_1, \gamma) \subseteq C(\alpha_2, \gamma), \forall \gamma \geq 0$. Take a location $x \in C(\alpha_1, \gamma)$. For every $y \in L$, if $d(u, y) < d(u, x) - \alpha_2$ then $d(u, y) < d(u, x) - \alpha_2 < d(u, x) - \alpha_1$. So,

$$|\{u \in U : d(y, u) < d(x, u) - \alpha_2\}| \leq |\{u \in U : d(y, u) < d(x, u) - \alpha_1\}| \leq \gamma n.$$

Therefore $x \in C(\alpha_2, \gamma)$.

Second we prove that $\gamma_1 < \gamma_2 \Rightarrow C(\alpha, \gamma_1) \subseteq C(\alpha, \gamma_2), \forall \alpha \geq 0$. Take $x \in C(\alpha, \gamma_1)$. Then, for every $y \in L$, $|\{u \in U : d(y, u) < d(x, u) - \alpha\}| \leq \gamma_1 n$. Thus,

$$|\{u \in U : d(y, u) < d(x, u) - \alpha\}| \leq \gamma_1 n < \gamma_2 n, \forall y \in L,$$

and $x \in C(\alpha, \gamma_2)$. □

The Condorcet locations are $C = C(0, 1/2)$. If there is no Condorcet location then both parameters, α and γ, can be used to relax the Condorcet conditions to allow that a solution exists. There is always a tolerance distance α and a qualified majority γ such that an [α,γ]-Condorcet solution exists. Take, for instance, $\alpha = \max\{d(u, x) : u \in U\}$, for any $x \in L$ or $\gamma = 1$.

The Simpson and Tolerant Condorcet locations are obtained by the minimum feasible relaxations using only one of the parameters. They came from the special cases of the [α,γ]-Condorcet locations for $\gamma = 1/2$ and $\alpha = 0$, respectively. Let

$$\alpha^* = \min\{\alpha \geq 0 : C(\alpha, 1/2) \neq \emptyset\}$$

and

$$\gamma^* = \min\{\gamma \geq 0 : C(0, \gamma) \neq \emptyset\}.$$

Then the **Tolerant Condorcet locations** [**?**] and the **Simpson locations** [**?**] are the [α^*,1/2]-Condorcet locations and the [0,γ^*]-Condorcet locations, respectively. Let $T = C(\alpha^*, 1/2)$ and $S = C(0, \gamma^*)$ denote the set of Tolerant Condorcet locations and the Simpson locations.

The minimum relaxations can be applied for any fixed value of the other parameter. So the α-Simpson locations and the γ-Tolerant Condorcet locations are defined as follows

Definition 4 *For a fixed* α*, the* α-**Simpson locations** *are the locations* $x \in S(\alpha) = C(\alpha, \gamma^*(\alpha))$ *where*

$$\gamma^*(\alpha) = \min\{\gamma \geq 0 : C(\alpha, \gamma) \neq \emptyset\}.$$

Definition 5 *For a fixed* γ*, the* γ**-Tolerant Condorcet locations** *are the locations* $x \in T(\gamma) = C(\alpha^*(\gamma), \gamma)$ *where*

$$\alpha^*(\gamma) = \min\{\alpha \geq 0 : C(\alpha, \gamma) \neq \emptyset\}.$$

These two parameters could be used, at the same time, to relax the Condorcet conditions and to locate the facility at an [α,γ]-Condorcet point for values $\alpha \geq 0$ and $\gamma \geq 0$. In order to be as close as possible to the Condorcet rules, both parameters should be small, leading to the two objective problem of minimizing α and γ subject to $C(\alpha, \gamma) \neq \emptyset$. The Pareto pairs of this two-objective problem will be called efficient pairs.

Definition 6 *Let* α *be a tolerance distance and* γ *be a qualified majority. The pair* (α, γ) *is* **efficient** *if* $C(\alpha, \gamma) \neq \emptyset$ *and* $C(\alpha', \gamma') = \emptyset$ *for every* $\alpha' \leq \alpha$*,* $\gamma' \leq \gamma$ *with* $(\alpha', \gamma') \neq (\alpha, \gamma)$*. A location* x *is an* **efficient Condorcet location** *if and only if* $x \in C(\alpha, \gamma)$ *for some efficient pair* (α, γ)*.*

Let E denote the set of efficient pairs and EC denote the set of efficient Condorcet locations. Then

$$EC = \bigcup_{(\alpha,\gamma)\in E} C(\alpha, \gamma).$$

Proposition 7 $(\alpha, \gamma) \in E$ *if and only if* $\alpha^*(\gamma) = \alpha$ *and* $\gamma^*(\alpha) = \gamma$*.*

Proof. Let $(\alpha_0, \gamma_0) \in E$. Then, by definition of E, $C(\alpha_0, \gamma_0) \neq \emptyset$ and $C(\alpha_0, \gamma) = \emptyset$, $\forall \gamma < \gamma_0$. Therefore $\gamma^*(\alpha_0) = \min\{\gamma : C(\alpha_0, \gamma) \neq \emptyset\} = \gamma_{0.}$ Again, by definition of E, $C(\alpha, \gamma_0) = \emptyset$, $\forall \alpha < \alpha_0$. Therefore $\alpha^*(\gamma_0) = \min\{\alpha : C(\alpha, \gamma_0) \neq \emptyset\} = \alpha_{0.}$

Conversely, if $\alpha^*(\gamma_0) = \alpha_0$ and $\gamma^*(\alpha_0) = \gamma_0$, take a pair (α, γ) with $\alpha \leq \alpha_0$ and $\gamma \leq \gamma_0$ but $(\alpha, \gamma) \neq (\alpha_0, \gamma_0)$. If $\alpha \neq \alpha_0$ then $\alpha < \alpha_0$ and $C(\alpha, \gamma) \subseteq C(\alpha, \gamma_0) = \emptyset$. If $\gamma \neq \gamma_0$ then $\gamma < \gamma_0$ and $C(\alpha, \gamma) \subseteq C(\alpha_0, \gamma) = \emptyset$. Therefore $(\alpha_0, \gamma_0) \in E$. $\square$

Thus, for every $(\alpha, \gamma) \in E$,

$$C(\alpha, \gamma) = C(\alpha, \gamma^*(\alpha)) = S(\alpha)$$

and

$$C(\alpha, \gamma) = C(\alpha^*(\gamma), \gamma) = T(\gamma).$$

Therefore, the efficient Condorcet locations are the points that are, at the same time, α-Simpson and γ-Tolerant Condorcet locations for some tolerance distance α and qualified majority γ related by $\alpha^*(\gamma) = \alpha$ and $\gamma^*(\alpha) = \gamma$. A tolerance distance α (or a qualified majority γ) is called **efficient** if it is part of an efficient pair $(\alpha, \gamma) \in E$. Let $E\Delta$ and $E\Gamma$ be the sets of the efficient tolerance distances and of the efficient qualified majorities, respectively. Then

$$\alpha \in E\Delta \Longleftrightarrow (\alpha, \gamma^*(\alpha)) \in E \Longleftrightarrow \alpha^*(\gamma^*(\alpha)) = \alpha$$

and

$$\gamma \in E\Gamma \Longleftrightarrow (\alpha^*(\gamma), \gamma) \in E \Longleftrightarrow \gamma^*(\alpha^*(\gamma)) = \gamma.$$

Thus

$$EC = \bigcup_{(\alpha,\gamma)\in E} C(\alpha, \gamma) = \bigcup_{\alpha \in E\Delta} S(\alpha) = \bigcup_{\gamma \in E\Gamma} T(\gamma).$$

A same location may appear in this set for different α, γ pairs. The set EC of efficient location in itslef is of no interest to the decision maker without additional information of the corresponding α, γ values. Let

$$ECP = \{(x, \alpha, \gamma) : x \in C(\alpha, \gamma)\}.$$

2 The solution procedure.

We consider the solution procedures only for the cases where U and L are finite sets, however the definitions can be applied to a non finite set L. All users are equally important but several users may be at the same point; the number of users at a point is interpreted as the weight of this user point. This point of view provides a more general weighted version of the problem.

This section is organized in four subsections. First, we consider the solution procedure for the [α,γ]-Condorcet location problem, that provide the set $C(\alpha, \gamma)$, for any fixed values for the parameters α and γ. Second, we provide the algorithm for getting the α-Simpson; i.e. the locations in $S(\alpha) = C(\alpha, \gamma^*(\alpha))$ for a fixed value for α. Third, we consider the algorithm for getting the γ-Tolerant Condorcet locations; i.e. the locations in $T(\gamma) =$

$C(\alpha^*(\gamma), \gamma)$, for a fixed value for γ. Finally, the procedure for determining the set EC of efficient Condorcet locations is given in the fourth subsection.

Let $L = \{x_1, x_2, ..., x_m\}$ be the set of possible locations for a facility for the set of users $U = \{u_1, u_2, ..., u_n\}$. Let the user-location distances be given by $d_{ki} = d(u_k, x_i)$, for $k = 1, 2, ..., n$ and $i = 1, 2, ..., m$, in an $n \times m$ matrix D. Let $\delta_{ij}^k = d_{kj} - d_{ki} = d(u_k, x_j) - d(u_k, x_i)$ be the comparison differences to be used in the algorithms.

2.1 The [α,γ]-Condorcet Problem

To solve the [α,γ]-Condorcet problem consists of obtaining the locations in $C(\alpha, \gamma)$, for given α and γ. The value γ is the qualified rejection majority and α is the tolerance distance for the preference of all the users. Then, for every user u_k, a location x_i is preferred to the location x_j if and only if $\delta_{ij}^k > \alpha$. The location x_j is rejected if there is a location x_i preferred by more than γn users. Therefore the location x_j is an [α,γ]-Condorcet location if and only if $\max_i |\{k : \delta_{ij}^k > \alpha\}| \leq \gamma n$. Thus the problem is solved by computing the score of each location by

$$R_j(\alpha) = \max_i |\{k : \delta_{ij}^k > \alpha\}|, \text{ for } j = 1, 2, ..., m$$

and comparing this value with γn.

Then, the [α,γ]-Condorcet locations are given by

$$C(\alpha, \gamma) = \{x_j : R_j(\alpha) \leq \gamma n\}.$$

The algorithm to get the [α,γ]-Condorcet locations is as follows.

Algorithm $C(\alpha, \gamma)$ for given α and γ.

1. For $i, j = 1, 2, ..., m$ compute $P_{ij}(\alpha) = |\{k : \delta_{ij}^k > \alpha\}|$.
2. For $j = 1, 2, ..., m$ compute $R_j(\alpha) = \max_i P_{ij}(\alpha)$.
3. The solution is $C(\alpha, \gamma) = \{x_j : R_j(\alpha) \leq \gamma n\}$.

Proposition 8 *For every tolerance distance α and qualified majority γ, the algorithm $C(\alpha, \gamma)$ takes $O(nm^2)$ time.*

Proof. To compute each $P_{ij}(\alpha)$ at step 1 we need $O(n)$ operations, and this must be repeated $O(m^2)$ times. Step 2 and 3 take $O(m^2)$ comparisons. Therefore the entire procedure takes $O(nm^2)$ operations. □

The traditional Condorcet locations are found by taking $\alpha = 0$ and $\gamma = 1/2$. Hence compute

$$R_j(0) = \max_i |\{k : \delta_{ij}^k > 0\}|, \text{ for } j = 1, 2, ..., m$$

and the Condorcet locations are given by

$$C = \{x_j : R_j(0) \leq n/2\}.$$

Corollary 9 *The Condorcet locations are obtained in* $O(nm^2)$.

2.2 The α-Simpson problem

To solve the α-Simpson problem consists of obtaining the locations in $S(\alpha)$, for a given α. The value α is the tolerance distance for the preference of the users. Since $S(\alpha) = C(\alpha, \gamma^*(\alpha))$, compute $r^*(\alpha) = \min_j R_j(\alpha)$ and take $\gamma^*(\alpha) = \frac{1}{n} r^*(\alpha)$. Then the α-Simpson locations are given by

$$S(\alpha) = \{x_j : R_j(\alpha) = r^*(\alpha)\}.$$

The algorithm for the α-Simpson locations is as follows.

Algorithm $S(\alpha)$ for a given α.

1. For $i, j = 1, 2, ..., m$ compute $P_{ij}(\alpha) = |\{k : \delta_{ij}^k > \alpha\}|$.
2. For $j = 1, 2, ..., m$ compute $R_j(\alpha) = \max_i P_{ij}(\alpha)$.
3. Compute $r^*(\alpha) = \min_j R_j(\alpha)$.
4. The α-Simpson locations are given by $S(\alpha) = \{x_j : R_j(\alpha) = r^*(\alpha)\}$.

Proposition 10 *For every tolerance distance* α*, the algorithm* $S(\alpha)$ *takes* $O(nm^2)$ *time.*

Proof. The algorithm $S(\alpha)$ is the same as the algorithm $C(\alpha,\gamma)$ except for the supplementary step 3 where $\gamma n = r^*(\alpha)$ is calculated in $O(m)$ time. $\square$

To get Simpson locations take $\alpha = 0$. The Simpson locations are obtained by computing

$$r^* = \min_j R_j(0)$$

and then $S = \{x_j : R_j(0) = r^*\}$. So $\gamma^* = r^*/n$. If $C = \emptyset$ then γ^* can be bigger than $1/2$ and if $C \neq \emptyset$ then it can be smaller than $1/2$.

Corollary 11 *The Simpson locations are obtained in $O(nm^2)$.*

2.3 The γ-Tolerant Problem

To solve the γ-Tolerant problem consists of getting the locations in $T(\gamma)$, for a given γ. The value γ is the qualified rejection majority. Since $T(\gamma) = C(\alpha^*(\gamma),\gamma)$ we need to obtain the tolerance distance $\alpha^*(\gamma)$. Take $\alpha = 0$ and apply the algorithm $C(\alpha,\gamma)$ to get $C(0,\gamma)$. If $C(0,\gamma) \neq \emptyset$ then $\alpha^*(\gamma) = 0$ and $T(\gamma) = C(0,\gamma)$. Otherwise increase the value of α until $C(\alpha,\gamma) \neq \emptyset$; i.e., until $r^*(\alpha) = \min_j R_j(\alpha) \leq \gamma n$. Then $\alpha^*(\gamma) = \min\{\alpha \geq 0 : r^*(\alpha) \leq \gamma n\}$ and $T(\gamma) = C(\alpha^*(\gamma),\gamma)$.

Note that when α increases, some functions $R_j(\alpha)$ decrease. The functions $R_j(\alpha)$ are piecewise constant dropping only at the values δ^k_{ij}; i.e. at the entries of the matrices $\Delta^k = [\delta^k_{ij}]$, $k = 1,..,n$. Namely, the value $R_j(\alpha)$ decreases for the locations x_j such that $\{i : \delta^k_{ij} = \alpha\} \neq \emptyset$. Then we only need to consider values for α in the set

$$\Delta = \{\delta^k_{ij} : i,j = 1,...,m; k = 1,...,n\}.$$

We may drop equal values from Δ. So

$$\alpha^*(\gamma) = \min\{\alpha \in \Delta : C(\alpha,\gamma) \neq \emptyset\} = \min\{\alpha \in \Delta : r^*(\alpha) \leq \gamma n\}.$$

To find $\alpha^*(\gamma)$, we need to apply the algorithm $C(\alpha,\gamma)$ to know if $C(\alpha,\gamma) = \emptyset$ for $\alpha \in \Delta$. Since the size of set Δ of relevant tolerance distances is $|\Delta| = O(nm^2)$, an exhaustive search would mean to apply $O(nm^2)$ times the algorithm $C(\alpha,\gamma)$. However, with a dichotomic search of $\alpha^*(\gamma)$ in Δ, the number of runs of the algorithm is reduced.

The proposed algorithm for finding the γ-Tolerant Condorcet locations, for any qualified majority γ is as follows.

Algorithm $T(\gamma)$ for a given γ.

1. Arrange the set of tolerance distances in Δ in increasing order; i.e. let $\Delta = [a_{(1)}, a_{(2)}, ..., a_{(k)}, ..., a_{(|\Delta|)}]$ such that $a_{(k)} \leq a_{(k+1)}$ for every k.
2. Apply the algorithm $C(\alpha, \gamma)$ for $\alpha = 0$ and the given γ to get the set of γ-Condorcet locations $C(0, \gamma)$.
3. If $C(0, \gamma) \neq \emptyset$ then take $\alpha^*(\gamma) \leftarrow 0$ and $T(\gamma) \leftarrow C(0, \gamma)$; stop. Otherwise go to step 4.
4. Take the Tolerant Condorcet candidates as $T \leftarrow C(a_{(|\Delta|)}, \gamma)$.
5. Initialize the below and above tolerance distance indices for the search by $k_1 \leftarrow 1$ and $k_2 \leftarrow |\Delta|$.
6. Take $k \leftarrow \lfloor (k_1 + k_2)/2 \rfloor$ and the tolerance distance $\alpha \leftarrow a_{(k)}$.
7. Apply the algorithm $C(\alpha, \gamma)$ for the current tolerance distance α and the given qualified majority γ.
8. If $C(\alpha, \gamma) = \emptyset$ then update the below index to k ($k_1 \leftarrow k$). Otherwise update the Tolerant Condorcet candidates T to $C(\alpha, \gamma)$ ($T \leftarrow C(\alpha, \gamma)$) and the above index to k ($k_2 \leftarrow k$).
9. If $k_1 + 1 < k_2$ then go to step 6. Otherwise take $\alpha^*(\gamma) \leftarrow \alpha$ and $T(\gamma) \leftarrow T$; stop.

Proposition 12 *For every qualified majority γ, the algorithm $T(\gamma)$ takes $O(nm^2(\log n + \log m))$ time.*

Proof. Since $|\Delta| = O(nm^2)$ the set Δ is arranged in increasing order in $O(|\Delta| \log |\Delta|) = O(nm^2(\log n + \log m))$ time. Therefore Step 1 requires $O(nm^2(\log n + \log m))$ operations. The number of times that the algorithm $C(\alpha, \gamma)$ is applied at step 7 is $O(\log |\Delta|) = O(\log n + \log m)$. Thus the algorithm $T(\gamma)$ takes $O(nm^2(\log n + \log m))$ time. $\square$

To get the Tolerant Condorcet locations take $\gamma = 1/2$ and apply the algorithm $T(\gamma)$ to get $T = T(1/2)$.

Corollary 13 *The Tolerant Condorcet problem is solved in $O(nm^2(\log n + \log m))$ time.*

2.4 The Efficient Condorcet Problem

To solve the efficient Condorcet problem consists of obtaining the locations in EC. The set of efficient Condorcet locations is obtained by

$$EC = \bigcup_{(\alpha,\gamma)\in E} C(\alpha, \gamma),$$

where $E = \{(\alpha, \gamma) : \alpha^*(\gamma) = \alpha \text{ and } \gamma^*(\alpha) = \gamma\}$. We have seen that for the function $\alpha^*(.)$ we only need to consider values for α in the finite set Δ. For the function $\gamma^*(.)$ we only need to consider values for γ in the finite set $\Gamma = \{k/n : k = 0, 1, ..., n\}$. Therefore for getting the set of efficient pairs (α, γ) we have to consider only the finite set of pairs in $\Delta \times \Gamma$. Then

$$E = \{(\alpha, \gamma) \in \Delta \times \Gamma : \alpha^*(\gamma) = \alpha \text{ and } \gamma^*(\alpha) = \gamma\}$$

The sizes of these sets are: $|\Delta| = O(nm^2)$ and $|\Gamma| = O(n)$. Therefore $|\Delta \times \Gamma| = O(n^2m^2)$.

Then the set of efficient Condorcet locations is obtained by solving $C(\alpha, \gamma)$ for the pairs $(\alpha, \gamma) \in \Delta \times \Gamma$ such that $\alpha^*(\gamma) = \alpha$ and $\gamma^*(\alpha) = \gamma$. The set of efficient tolerance distances and the set of efficient qualified majorities are the projections of the set E given by $E\Delta = \{\alpha \in \Delta : (\alpha, \gamma) \in E$, for some $\gamma\}$ and $E\Gamma = \{\gamma \in \Gamma : (\alpha, \gamma) \in E$, for some $\alpha\}$. Then

$$E = \{(\alpha, \gamma^*(\alpha)) : \alpha \in E\Delta\} = \{(\alpha^*(\gamma), \gamma) : \gamma \in E\Gamma\}$$

and the set EC of efficient Condorcet locations can also be obtained by

$$EC = \bigcup_{(\alpha,\gamma)\in E} C(\alpha, \gamma) = \bigcup_{\alpha\in E\Delta} C(\alpha, \gamma^*(\alpha)) = \bigcup_{\alpha\in E\Delta} S(\alpha)$$

and by

$$EC = \bigcup_{(\alpha,\gamma)\in E} C(\alpha, \gamma) = \bigcup_{\alpha\in E\Gamma} C(\alpha^*(\gamma), \gamma) = \bigcup_{\alpha\in E\Gamma} T(\gamma).$$

Thus to get the set EC we can solve the α-Simpson problems for all the values α in $E\Delta$ or the γ-Tolerant Condorcet problems for all the values γ in $E\Gamma$.

Since the sets $C(\alpha, \gamma)$ are non decreasing with respect to both parameters, then $\alpha^*(.)$ and $\gamma^*(.)$ are nonincreasing functions. Moreover, since $\alpha^*(.)$ and

$\gamma^*(.)$ only take values in the finite sets Δ and Γ they are stepwise functions and the efficient tolerance distances and qualified majorities are at the steps of these functions. They are easily found by arranging the values of the sets Δ and Γ in decreasing order.

Proposition 14 *Let $\alpha_{(i)}$ and $\gamma_{(i)}$ denote the i-th value of Δ and Γ in decreasing order; i.e., such that $\alpha_{(i+1)} < \alpha_{(i)}$ and $\gamma_{(i+1)} < \gamma_{(i)}$, for every i.*

a) *If $\alpha^*(\gamma_{(i)}) < \alpha^*(\gamma_{(i+1)})$ then $(\alpha^*(\gamma_{(i)}), \gamma_{(i)}) \in E$; i.e., $\gamma_{(i)} \in E\Gamma$.*

b) *If $\gamma^*(\alpha_{(i)}) < \gamma^*(\alpha_{(i+1)})$ then $(\alpha_{(i)}, \gamma^*(\alpha_{(i)})) \in E$; i.e., $\alpha_{(i)} \in E\Delta$.*

Proof. First, if $\alpha^*(\gamma_{(i)}) < \alpha^*(\gamma_{(i+1)})$ then $\alpha^*(\gamma_{(i)}) < \alpha^*(\gamma')$, for any $\gamma' < \gamma_{(i)}$. Thus $\gamma^*(\alpha^*(\gamma_{(i)})) = \gamma_{(i)}$ and then $(\alpha^*(\gamma_{(i)}), \gamma_{(i)}) \in E$.

In the same way, from $\gamma^*(\alpha_{(i)}) < \gamma^*(\alpha_{(i+1)})$ we get $\gamma^*(\alpha_{(i)}) < \gamma^*(\alpha')$, for any $\alpha' < \alpha_{(i)}$. Therefore $\alpha^*(\gamma^*(\alpha_{(i)})) = \alpha_{(i)}$ and $(\alpha_{(i)}, \gamma^*(\alpha_{(i)})) \in E$. □

Since only values for γ in $\Gamma = \{k/n : k = 0, 1, ..., n\}$ have to be considered, the following result gives a way to get the finite set of efficient qualified majorities $E\Gamma$.

Proposition 15 *$\frac{k}{n} \in E\Gamma$ if and only if*

$$k = 0 \text{ or } C\left(\alpha^*\left(\frac{k}{n}\right), \frac{k-1}{n}\right) = \emptyset.$$

Proof. Note that if $\gamma_{(i)} = \frac{k}{n}$ then $\gamma_{(i+1)} = \frac{k-1}{n}$. Thus, $C\left(\alpha^*\left(\frac{k}{n}\right), \frac{k-1}{n}\right) = \emptyset$ if and only if $\alpha^*\left(\frac{k}{n}\right) < \alpha^*\left(\frac{k-1}{n}\right)$. Therefore this condition is equivalent to

$$\left(\alpha^*\left(\frac{k}{n}\right), \frac{k}{n}\right) \in E.$$

and to $\frac{k}{n} \in E\Gamma$. □

Then the set EC can be obtained by using

$$EC = \bigcup_{k/n \in E\Gamma} C\left(\alpha^*\left(\frac{k}{n}\right), \frac{k}{n}\right) = \bigcup_{k/n \in E\Gamma} T\left(\frac{k}{n}\right)$$

On the other hand, the finite set Δ of values for the tolerance distance provides another ways for obtaining EC using $E\Delta$. Note that since $\alpha_{(i)}$ denotes the i-th value of Δ in decreasing order, we have $\alpha_{(i+1)} < \alpha_{(i)}$ for every i. Therefore, similarly as in proposition 12

$$E\Delta = \{\alpha_{(i)} \in \Delta : C(\alpha_{(i+1)}, \gamma^*(\alpha_{(i)})) = \emptyset\}.$$

Then the set EC can be obtained by using

$$EC = \bigcup_{\alpha_{(i)} \in E\Delta} C(\alpha_{(i)}, \gamma^*(\alpha_{(i)})) = \bigcup_{\alpha_{(i)} \in E\Delta} S(\alpha_{(i)}).$$

There are also two ways for getting an efficient Condorcet location. First by taking an arbitrary α and getting $\gamma^*(\alpha)$ and $\alpha^*(\gamma^*(\alpha))$. Then

$$(\alpha^*(\gamma^*(\alpha)), \gamma^*(\alpha)) \in E$$

and

$$C(\alpha^*(\gamma^*(\alpha)), \gamma^*(\alpha)) = S(\alpha^*(\gamma^*(\alpha))) = T(\gamma^*(\alpha)) \subset EC.$$

And second by taking an arbitrary γ and getting $\alpha^*(\gamma)$ and $\gamma^*(\alpha^*(\gamma))$. Then $(\alpha^*(\gamma), \gamma^*(\alpha^*(\gamma))) \in E$ and

$$C(\alpha^*(\gamma), \gamma^*(\alpha^*(\gamma))) = S(\alpha^*(\gamma)) = T(\gamma^*(\alpha^*(\gamma))) \subset EC.$$

These relations are directly obtained from the definitions of $S(\alpha), T(\gamma), \alpha^*(\gamma)$ and $\gamma^*(\alpha)$.

Finally, to get the set EC of the efficient Condorcet locations we use

$$EC = \bigcup_{\gamma \in E\Gamma} T(\gamma) = \bigcup_{\gamma \in E\Gamma} C(\alpha^*(\gamma), \gamma)$$

and

$$E\Gamma = \{0\} \cup \left\{ \frac{k}{n} : C\left(\alpha^*\left(\tfrac{k}{n}\right), \tfrac{k-1}{n}\right) = \emptyset \right\}.$$

Therefore

$$EC = T(0) \cup \bigcup_{k=1}^{n} \left\{ T\left(\tfrac{k}{n}\right) : C\left(\alpha^*\left(\tfrac{k}{n}\right), \tfrac{k-1}{n}\right) = \emptyset \right\}.$$

Thus, the set EC can be obtained in $O(n^2 m^2 (\log n + \log m))$ time by solving $O(n)$ γ-Tolerant Condorcet problems for increasing values in Γ.

Algorithm EC.

1. Apply the algorithm $S(\alpha)$ for $\alpha = 0$ to get $\gamma^* = \gamma^*(0)$.
2. Apply the algorithm $T(\gamma)$ for $\gamma = 0$ to get $T(0)$ and $\alpha^* = \alpha^*(0)$.
3. Initialize $EC = T(0)$ and $E = \{(\alpha^*, 0)\}$.
4. Take $k \leftarrow 0$.
5. Do $k \leftarrow k + 1$.
6. Apply the algorithm $T(\gamma)$ for $\gamma = k/n$ to get $T = T(k/n)$.
7. Take $a \leftarrow \alpha^*(k/n)$.
8. Apply the algorithm $C(\alpha, \gamma)$ for $\alpha = a$ and $\gamma = (k-1)/n$.
9. If $C(\alpha, \gamma) = \emptyset$ then do $EC \leftarrow EC \cup T$ and $E \leftarrow E \cup \{(a, k/n)\}$.
10. If $k/n \leq \gamma^*$ then go to step 5. Otherwise stop.

Proposition 16 *The algorithm EC takes $O(n^2m^2(\log n + \log m))$ time.*

Proof. The algorithms $S(\alpha)$ and $T(\gamma)$ are applied once at steps 1 and 2. Since step 5 is done $|\Gamma| = O(n)$ times, the algorithms $T(\gamma)$ and $C(\alpha, \gamma)$ are applied $O(n)$ times. These algorithms are $O(nm^2(\log n + \log m))$ and $O(nm^2)$, respectively, so the number of operations in steps 6 and 7 are $O(n^2m^2(\log n + \log m))$ and $O(n^2m^2)$. Thus, to provide the set EC, the algorithm takes $O(n^2m^2(\log n + \log m))$ time. □

The Tolerant Condorcet locations can be obtained by testing, in the step 8, the first time that $k/n \leq 1/2$.

3 An example.

Consider the location of a point in the network shown in figure 1.

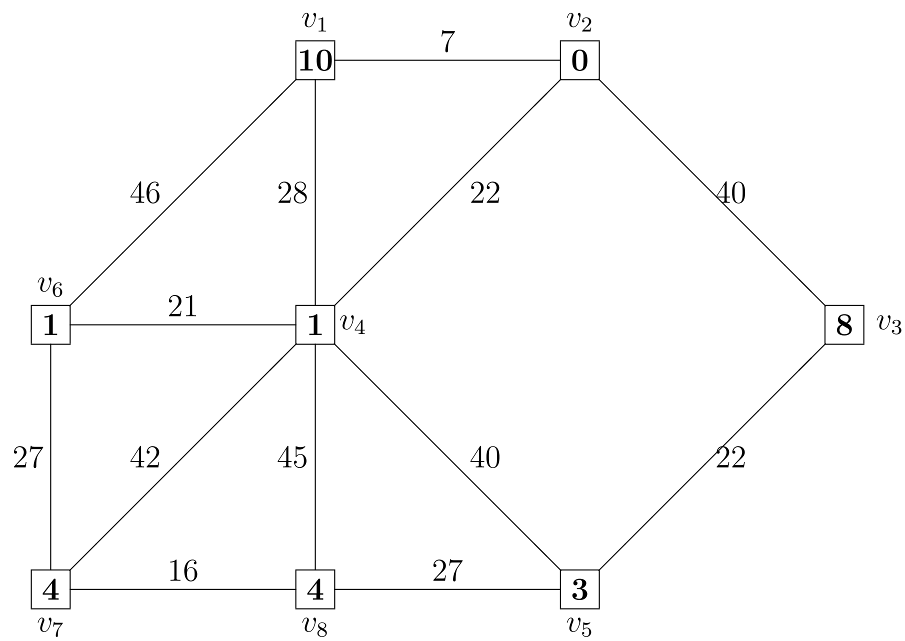

Figure 1. The example.

The possible location of the facility is restricted to the vertices of the network. Then $L = \{v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_8\}$. Consider 31 users at the vertices of the network graph where the number of users at every vertex is given by

Vertices	v_1	v_2	v_3	v_4	v_5	v_6	v_7	v_8
Number of users	10	0	8	1	3	1	4	4

Let $D = [d_{ij}]$ denote the distance matrix between vertices; i.e., $d_{ij} = d(v_i, v_j)$. Then

$$D = \begin{bmatrix} 0 & 7 & 47 & 28 & 68 & 46 & 70 & 73 \\ 7 & 0 & 40 & 22 & 62 & 43 & 64 & 67 \\ 47 & 40 & 0 & 62 & 22 & 83 & 65 & 49 \\ 28 & 22 & 62 & 0 & 40 & 21 & 42 & 45 \\ 68 & 62 & 22 & 40 & 0 & 61 & 43 & 27 \\ 46 & 43 & 83 & 21 & 61 & 0 & 27 & 43 \\ 70 & 64 & 65 & 42 & 43 & 27 & 0 & 16 \\ 73 & 67 & 49 & 45 & 27 & 43 & 16 & 0 \end{bmatrix}$$

To get the Condorcet solution compute the number of users that prefer a vertex v_i to a vertex v_j that are given by

$$P_{ij} = |\{u : d(u, v_j) - d(u, v_i) > 0\}| = \sum_{d_{kj} > d_{ki}} u(k),$$

where $u(k)$ is the number of users at vertex v_k. For instance,

$$P_{12} = |\{u : d(u, v_2) - d(u, v_1) > 0\}| = \sum_{d_{k2} > d_{k1}} u(k) = u(1) = 10,$$

$$P_{13} = |\{u : d(u, v_3) - d(u, v_1) > 0\}| = u(1) + u(2) + u(4) + u(6) = 12,$$

and so on.

Then the matrix P is

$$P = \begin{bmatrix} 0 & 10 & 12 & 18 & 12 & 18 & 19 & 19 \\ 21 & 0 & 16 & 18 & 12 & 18 & 19 & 19 \\ 19 & 15 & 0 & 11 & 18 & 11 & 21 & 21 \\ 13 & 13 & 20 & 0 & 16 & 22 & 23 & 12 \\ 19 & 19 & 13 & 15 & 0 & 15 & 22 & 22 \\ 13 & 13 & 20 & 9 & 16 & 0 & 12 & 12 \\ 12 & 12 & 10 & 8 & 9 & 19 & 0 & 16 \\ 12 & 11 & 10 & 19 & 9 & 19 & 15 & 0 \end{bmatrix}$$

So the values of the scores $r_j = R_j(0) = \max_i P_{ij}$ are given by

$$R(0) = \begin{bmatrix} 21 & 19 & 20 & 19 & 18 & 22 & 23 & 22 \end{bmatrix}$$

The Simpson location is v_5 with $r_5 = 18$. Thus $r^* = 18$ and $\gamma^* = 18/31 > 1/2$, so there is no Condorcet location.

For an arbitrary α we need to use the matrix with the number of users that prefer a location to another one taking into account the tolerance distance α; i.e., $P(\alpha)$ instead of P where

$$P_{ij}(\alpha) = |\{u : \delta_{ij}^k = d_{kj} - d_{ki} > \alpha\}| = \sum_{\delta_{ij}^k > \alpha} u(k).$$

For $\alpha = 1$ we have

$$P(\alpha) = P(1) = \begin{bmatrix} 0 & 10 & 12 & 18 & 12 & 18 & 19 & 19 \\ 21 & 0 & 12 & 18 & 12 & 18 & 19 & 19 \\ 19 & 15 & 0 & 11 & 18 & 11 & 21 & 21 \\ 13 & 13 & 20 & 0 & 12 & 22 & 23 & 12 \\ 19 & 19 & 13 & 15 & 0 & 15 & 22 & 22 \\ 13 & 9 & 10 & 9 & 16 & 0 & 12 & 12 \\ 12 & 12 & 10 & 8 & 9 & 19 & 0 & 16 \\ 12 & 11 & 10 & 19 & 9 & 19 & 15 & 0 \end{bmatrix}$$

Note that from $P = P(0)$ to $P(1)$ only four values are modified. For $0 \leq \alpha < 3$ the matrix $P(\alpha)$ is modified at $\alpha = 1$ and at $\alpha = 2$ but the scores $R(\alpha)$ do not change. By increasing α up to $\alpha = 3$ the vector of scores only changes in $R_1(\alpha)$ and $R_7(\alpha)$ that get 20 and 21, but the minimum is also at v_5 with $R_5(3) = 18$. For $\alpha = 3$ we have

$$P(\alpha) = P(3) = \begin{bmatrix} 0 & 10 & 12 & 18 & 12 & 18 & 19 & 11 \\ 20 & 0 & 12 & 18 & 12 & 18 & 19 & 19 \\ 19 & 15 & 0 & 11 & 18 & 11 & 21 & 21 \\ 13 & 13 & 20 & 0 & 12 & 22 & 12 & 12 \\ 19 & 19 & 13 & 15 & 0 & 15 & 11 & 22 \\ 13 & 9 & 10 & 5 & 16 & 0 & 12 & 12 \\ 12 & 12 & 10 & 8 & 9 & 19 & 0 & 5 \\ 11 & 11 & 10 & 19 & 9 & 19 & 15 & 0 \end{bmatrix}$$

The scores $R(3)$ are

$$R(3) = \begin{bmatrix} 20 & 19 & 20 & 19 & 18 & 22 & 21 & 22 \end{bmatrix}$$

With $\alpha = 4$ the scores are

$$R(4) = \left[\begin{array}{cccccccc} 20 & 19 & 16 & 19 & 18 & 22 & 21 & 22 \end{array}\right]$$

Then the only α-Simpson point is v_3 with majority $r^*(4) = 16 > n/2$. This is the only α-Simpson point up to $\alpha = 17$. For this value

$$R(17) = \left[\begin{array}{cccccccc} 19 & 19 & 16 & 18 & 18 & 22 & 21 & 18 \end{array}\right]$$

For $\alpha = 18$, the scores are

$$R(18) = \left[\begin{array}{cccccccc} 19 & 13 & 16 & 18 & 18 & 18 & 21 & 18 \end{array}\right]$$

Here we get the first time a value $R_j(\alpha) \leq 31/2$. Therefore, the tolerance distance is 18 with $\gamma^*(18) = 13/31 < 1/2$; the only Tolerant Condorcet point is v_2.

For $\alpha = 19$ the value of $\gamma^*(19)$ is also 13/31 but this minimum value is reached at v_2 and v_3. The procedure can continue until $\alpha = 62$ where $\gamma^*(62) = 0$ is reached at v_4; note that

$$\min_i \max_j d(v_i, v_j) = \max_j d(v_4, v_j) = 62.$$

This means that if the indifference threshold is 62 then the location v_4 is not rejected by any user and no other location verifies this with a smaller tolerance distance. Note v_4 is the vertex-center of the users.

Finally we have

$$r^*(\alpha) = \begin{cases} 18 & \text{for} \quad \alpha \in [0, 4) \\ 16 & \text{for} \quad \alpha \in [4, 18) \\ 13 & \text{for} \quad \alpha \in [18, 20) \\ 11 & \text{for} \quad \alpha \in [20, 38) \\ 8 & \text{for} \quad \alpha \in [38, 49) \\ 5 & \text{for} \quad \alpha \in [49, 51) \\ 4 & \text{for} \quad \alpha \in [51, 62) \\ 0 & \text{for} \quad \alpha \geq 62 \end{cases}$$

and

$$\alpha^*(r) = \begin{cases} 62 & \text{for} \quad r \in [0,4) \\ 51 & \text{for} \quad r \in [4,5) \\ 49 & \text{for} \quad r \in [5,8) \\ 38 & \text{for} \quad r \in [8,11) \\ 20 & \text{for} \quad r \in [11,13) \\ 18 & \text{for} \quad r \in [13,16) \\ 4 & \text{for} \quad r \in [16,18) \\ 0 & \text{for} \quad r \in [18,31] \end{cases}$$

The pairs of efficient pairs (α, r) are

$$E = \{(0,18),(4,16),(18,13),(20,11),(38,8),(49,5),(51,4),(62,0)\}.$$

Figure 2 shows the functions $\alpha^*(.)$ and $r^*(.)$ in this example. The efficient points are indicated by dots.

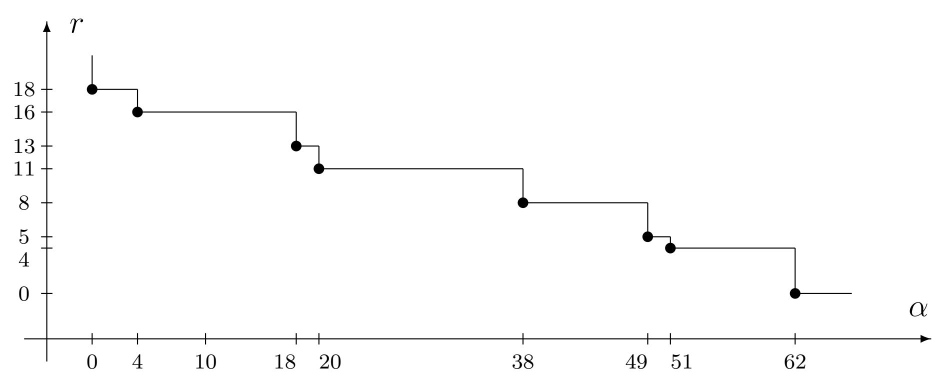

Figure 2. Functions $\alpha^*(.)$ and $r^*(.)$.

The corresponding $[\alpha,\gamma]$-Condorcet locations are

$$C(0,18) = \{v_5\}, \quad C(4,16) = \{v_3\}, \quad C(18,13) = \{v_2\},$$
$$C(20,11) = \{v_2\}, \quad C(38,8) = \{v_2\}, \quad C(49,5) = \{v_3\},$$
$$C(51,4) = \{v_2\}, \quad \text{and} \quad C(62,0) = \{v_4\}.$$

In addition to the Simpson location v_5 and the Tolerant Condorcet location v_2, there are two efficient Condorcet locations that are v_3 and the vertex-center v_4. Thus

$$EC = \{v_2, v_3, v_4, v_5\}.$$

4 Conclusions.

The distance matrix could be interpreted and/or replaced by any criterion-value matrix for a general decision point of view.

The voting location looks for compromise solutions among users. The Condorcet solution states that this compromise must consist of the location such that the majority of users does not disagree with it. However this compromise solution does not always exists. Then the Simpson proposal consists of minimizing the number of voters that disagree with the solution. The Tolerant proposal consists of minimizing the amount of disagreement for the majority of users. Here we show how these two ideas can be combined to get good compromise solutions and the corresponding polynomial algorithms to find them in the finite case of a single facility.

Since the notion of Condorcet solution has mainly been studied in a location theory context, we use the same terminology and a network to introduce these new notions. However, no particular features typical for location theory need to be used; almost everything can be seen as pure decision theory. In order to apply the results and algorithms in multicriteria decision or group decision making, we need normalized criteria or uniform value functions. The introduced notions are theoretical but this theory may be put into practice. It could be a theoretical development capable of being used in the foreseeable future; for instance in Location Decision Support Systems.

The situations where this methodology could be applied, are those location decisions that involve several groups of users whose opinion could be relevant. Politicians can be persuaded to realize that users will show indifference between two locations if the difference between them is not big enough. The practical meaning of the tolerance distance α would be the threshold for the difference of the distance to two possible locations to be understood as relevant by the users. The politicians could also determine the number of users that constitute a big enough majority against the proposed location to be rejected. So the rejection majority γ is obtained. Then the politician would choose the best location, from economical point of view, among the $[\alpha,\gamma]$-Condorcet locations.

Two lines of research from these results have been already opened by the authors. The first one consists of extending the results and algorithms for continuous sets on networks of possible locations as done by Hansen and Labbé [?] for the Simpson and Condorcet locations and by the authors [?] for the Tolerant Condorcet location. The second extension, that consists of considering multiple facilities, has not yet been attempted..

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