Les Cahiers du GERAD ISSN: 0711–2440

Variable Neighborhood Search: Methods and Applications

P. Hansen, N. Mladenović,
J.A. Moreno Pérez

G–2008–39

May 2008

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Pierre Hansen

GERAD and HEC Montréal
Montréal (Québec) Canada, H3T 2A7
pierre.hansen@gerad.ca

Nenad Mladenović

GERAD and School of Mathematics
Brunel University
UXbridge, United Kingdom
nenad.mladenovic@brunel.ac.uk

José A. Moreno Pérez

IUDR, University of La Laguna
La Laguna, Spain
jamoreno@ull.es

May 2008

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Abstract

Variable neighborhood search (VNS) is a metaheuristic, or framework for building heuristics, based upon systematic change of neighborhoods both in a decent phase to find a local minimum, and in a perturbation phase to get out of the corresponding valley. It was first proposed in 1997 and since has rapidly developed both in its methods and its applications. Both of these aspects are thouroughly reviewed and a large bibliography is given.

Key Words: Variable Neighborhood Search, Metaheuristic, Heuristic, Neighborhood.

Résumé

La Recherche à Voisinage Variable (RVV) est une métaheuristique, ou un cadre général pour la construction d'heuristiques, basée sur un changement systématique de voisinages à la fois dans la phase de descente pour trouver un minimum local, et dans la phase de perturbation pour sortir de la vallée correspondante. Elle a d'abord été proposée en 1997, et depuis, a été rapidement developpée tant dans ses méthodes que dans ses applications. Ces deux aspects sont passés en revue et une large bibliographie est donnée.

Mots clés : Recherche à Voisinage Variable, Métaheuristique, Heuristique, Voisinage.

1 Introduction

Variable neighborhood search (VNS) is a metaheuristic, or framework for building heuristics, aimed at solving combinatorial and global optimization problems. Its basic idea is systematic change of neighborhood combined with a local search. Since its inception, VNS has undergone many developments and been applied in numerous fields. We review below the basic rules of VNS and of its main extensions. Moreover, some of the most successful applications are briefly summarized. Pointers to many other ones are given in the reference list.

A deterministic optimization problem may be formulated as

$$\min\{f(x)|x \in X, X \subseteq \mathcal{S}\}, \tag{1}$$

where $\mathcal{S}, X, x$ and f denote respectively the *solution space* and *feasible set*, a *feasible solution* and a real-valued *objective function*, respectively. If $\mathcal{S}$ is a finite but large set a *combinatorial optimization* problem is defined. If $\mathcal{S} = R^n$, we talk about *continuous optimization*. A solution $x^* \in \mathcal{S}$ is *optimal* if

$$f(x^*) \leq f(x), \ \forall x \in \mathcal{S};$$

an *exact algorithm* for problem (1), if one exists, finds an optimal solution x^*, together with the proof of its optimality, or shows that there is no feasible solution, i.e., $\mathcal{S} = \emptyset$. Moreover, in practice, the time to do so should be finite (and not too large); if one deals with a continuous function one must admit a degree of tolerance i.e., stop when a feasible solution x^* has been found such that

$$f(x^*) < f(x) + \varepsilon, \ \forall x \in \mathcal{S} \ \mathit{or} \ \frac{f(x^*) - f(x)}{f(x^*)} < \varepsilon, \ \forall x \in \mathcal{S}$$

for some small positive ε.

Many practical instances of problems of the form (1), arising in Operations Research and other fields, are too large for an exact solution to be found in reasonable time. It is well-known from complexity theory [Garey and Johnson, 1978, Papadimitriou, 1994] that thousands of problems are *NP-hard*, that no algorithm with a number of steps polynomial in the size of the instances is known for solving any of them and that finding one would entail obtaining one for each and all of them. Moreover, in some cases where a problem admits a polynomial algorithm, the power of this polynomial may be so large that realistic size instances cannot be solved in reasonable time in worst case, and sometimes also in average case or most of the time.

So one is often forced to resort to *heuristics*, which yield quickly an approximate solution, or sometimes an optimal solution but without proof of its optimality. Some of these heuristics have a worst-case guarantee, i.e., the solution x_h obtained satisfies

$$\frac{f(x_h) - f(x)}{f(x_h)} \leq \varepsilon, \ \forall x \in X \tag{2}$$

for some ε, which is however rarely small. Moreover, this ε is usually much larger than the error observed in practice and may therefore be a bad guide in selecting a heuristic. In addition to avoiding excessive computing time, heuristics address another problem: local optima. A local optimum x_L of (1) is such that

$$f(x_L) \leq f(x), \ \forall x \in N(x_L) \cap X \tag{3}$$

where $N(x_L)$ denotes a neighborhood of x_L (ways to define such a neighborhood will be discussed below). If there are many local minima, the range of values they span may be large. Moreover, the globally optimum value $f(x^*)$ may differ substantially from the average value of a local minimum, or even from the best such value among many, obtained by some simple heuristic such as multistart (a phenomenon called the Tchebycheff catastrophe in Baum [1986]). There are, however, many ways to get out of local optima and, more precisely, the valleys which contain them (or set of solutions from which the descent method under consideration leads to them).

Metaheuristics are general frameworks to build heuristics for combinatorial and global optimization problems. For discussion of the best-known of them the reader is referred to the books of surveys [Reeves, 1993, Glover and Kochenberger, 2003, Burke and Kendall, 2005]. Some of the many successful applications of metaheuristics are also mentioned there.

Variable Neighborhood Search (VNS) [Mladenović and Hansen, 1997, Hansen and Mladenović, 1999, 2001a, 2003] is a metaheuristic which exploits systematically the idea of neighborhood change, both in descent to local minima and in escape from the valleys which contain them. VNS relies heavily upon the following observations:

Fact 1 *A local minimum with respect to one neighborhood structure is not necessary so for another;*

Fact 2 *A global minimum is a local minimum with respect to all possible neighborhood structures;*

Fact 3 *For many problems local minima with respect to one or several neighborhoods are relatively close to each other.*

This last observation, which is empirical, implies that a local optimum often provides some information about the global one. This may for instance be several variables with the same value in both. However, it is usually not known which ones are such. An organized study of the neighborhood of this local optimum is therefore in order, until a better one is found.

Unlike many other metaheuristics, the basic schemes of VNS and its extensions are simple and require few, and sometimes no parameters. Therefore in addition to providing very good solutions, often in simpler ways than other methods, VNS gives insight into the reasons for such a performance, which in turn can lead to more efficient and sophisticated implementations.

2 Background

VNS embeds a local search heuristic for solving combinatorial and global optimization problems. There are predecessors of this idea. It allows change of the neighborhood structures within this search. In this section we give a brief introduction into the variable metric algorithm for solving continuous convex problems and local search heuristics for solving combinatorial and global optimization problems.

2.1 Variable metric method

The variable metric method for solving unconstrained continuous optimization problem (1) has been suggested by Davidon [1959] and Fletcher and Powell [1963]. The idea is to change the metric (and thus the neighborhood) in each iteration such that the search direction (steepest descent with respect to the current metric) adapts better to the local shape of the function. In the first iteration a Euclidean unit ball in n dimensional space is used and the steepest descent (anti-gradient) direction found; in the next iterations, ellipsoidal balls are used and the steepest descent direction with respect to a new metric obtained after a linear transformation. The purpose of such changes is to built up, iteratively, a good approximation to the inverse of the Hessian matrix A^{-1} of f, that is, to construct a sequence of matrices H_i with the property,

$$\lim_{i \leftarrow \infty} H_i = A^{-1}.$$

In the convex quadratic programming case the limit is achieved after n iterations instead of ∞. In that way the so-called Newton search direction is obtained. The advantages are: (i) it is not necessary to find the inverse of the Hessian (which requires $O(n^3)$ operations) in each iteration; (ii) the second order information is not demanded. Assume that the function $f(x)$ is approximated by its Taylor series

$$f(x) = \frac{1}{2} x^T A x - b^T x \tag{4}$$

with positive definite matrix A ($A > 0$). Applying the first order condition $\nabla f(x) = Ax - b = 0$ we have $Ax_{opt} = b$, where x_{opt} is a minimum point. At the current point we have $Ax_i = \nabla f(x_i) + b$. We won't rigorously derive here the Davidon-Fletcher-Powell (DFP) algorithm for taking H_i into H_{i+1}. Let us just mention that subtracting these two equations and multiplying (from the left) by the inverse matrix A^{-1}, we have

$$x_{opt} - x_i = -A^{-1} \nabla f(x_i).$$

Subtracting the latest equation at x_{i+1} from the same equation at x_i gives

$$x_{i+1} - x_i = -A^{-1}(\nabla f(x_{i+1}) - \nabla f(x_i)). \tag{5}$$

Having made the step from x_i to x_{i+1}, we might reasonably want to require that the new approximation H_{i+1} satisfies (5) as if it were actually A^{-1}, that is,

$$x_{i+1} - x_i = -H_{i+1}(\nabla f(x_{i+1}) - \nabla f(x_i)). \tag{6}$$

We might also assume that the updating formula for matrix H_i should be of the form $H_{i+1} = H_i + U$, where U is a correction. It is possible to get different updating formulas for U and thus for H_{i+1}, keeping H_{i+1} positive definite ($H_{i+1} > 0$). In fact, there exists a whole family of updates, the Broyden family. From practical experience the so-called BFGS method seem to be most popular (see e.g. Gill et al. [1981] for details). Steps are listed in Algorithm 1.

From the above one can conclude that even in solving a convex program a change of metric, and thus change of the neighborhoods induced by that metric, may produce more efficient algorithms. Thus, using the idea of neighborhood change for solving NP-hard problems, could well lead to even greater benefits.

2.2 Local search

A *local search* heuristic consists of choosing an initial solution x, finding a direction of descent from x, within a neighborhood $N(x)$, and moving to the minimum of $f(x)$ within $N(x)$ along that direction; if there is no direction of descent, the heuristic stops, and otherwise it is iterated. Usually the steepest descent direction, also referred to as *best improvement*, is used.

Function `VarMetric(x);`
let $x \in R^n$ be an initial solution;
$H \leftarrow I$; $g \leftarrow -\nabla f(x)$;
for $i = 1$ `to` n **do**
  $\alpha^* \leftarrow arg\min_\alpha f(x + \alpha \cdot Hg)$;
  $x \leftarrow x + \alpha^* \cdot Hg$; $g \leftarrow -\nabla f(x)$;
  $H \leftarrow H + U$;
end

Algorithm 1: Variable metric algorithm

Function `BestImprovement(x);`
repeat
  $x' \leftarrow x$;
  $x \leftarrow arg\min_{y \in N(x)} f(y)$
until $(f(x) \geq f(x'))$;

Algorithm 2: Best improvement (steepest descent) heuristic

This set of rules is summarized in Algorithm 2, where we assume that an initial solution x is given. The output consists of a local minimum, also denoted with x, and its value. Observe that a neighborhood structure $N(x)$ is defined for all $x \in X$; in discrete optimization problems it usually consists of all vectors obtained from x by some simple modification, e.g. complementing one or two components of a 0-1 vector. Then, at each step, the neighborhood $N(x)$ of x is explored completely. As this may be time-consuming, an alternative is to use the *first descent* heuristic. Vectors $x_i \in N(x)$ are then enumerated systematically and a move is made as soon as a descent direction is found. This is summarized in Algorithm 3.

Function `FirstImprovement(x);`
repeat
  $x' \leftarrow x$; $i \leftarrow 0$;
  repeat
    $i \leftarrow i + 1$;
    $x \leftarrow arg\min\{f(x), f(x_i)\}$, $x_i \in N(x)$
  until $(f(x) < f(x_i)$ `or` $i = |N(x)|)$;
until $(f(x) \geq f(x'))$;

Algorithm 3: First improvement heuristic

3 Basic Schemes

Let us denote with $\mathcal{N}_k$, $(k = 1, \ldots, k_{max})$, a finite set of pre-selected neighborhood structures, and with $\mathcal{N}_k(x)$ the set of solutions in the k^{th} neighborhood of x. We will also use notation $\mathcal{N}'_k, k = 1, \ldots, k'_{max}$, when describing local descent. Neighborhoods $\mathcal{N}_k$ or $\mathcal{N}'_k$ may be induced from one or more metric (or quasi-metric) functions introduced into a solution space S. An *optimal solution* x_{opt} (or global minimum) is a feasible solution where a minimum of (1) is reached. We call $x' \in X$ a *local minimum* of (1) with respect to $\mathcal{N}_k$ (w.r.t. $\mathcal{N}_k$ for short), if there is no solution $x \in \mathcal{N}_k(x') \subseteq X$ such that $f(x) < f(x')$.

In order to solve (1) by using several neighborhoods, facts 1 to 3 can be used in three different ways: (i) deterministic; (ii) stochastic; (iii) both deterministic and stochastic. We first give in Algorithm 4 steps of the neighborhood change function that will be used later. Function `neighborhoodChange()` compares the new value $f(x')$ with the incumbent value

Function `NeighborhoodChange` (x, x', k);
1 **if** $f(x') < f(x)$ **then**
2 $\quad x \leftarrow x'$; $k \leftarrow 1$ /* Make a move */;
else
3 $\quad k \leftarrow k + 1$ /* Next neighborhood */ ;
end

Algorithm 4: Neighborhood change or Move or not function

$f(x)$ obtained in the neighborhood k (line 1). If an improvement is obtained, k is returned to its initial value and the new incumbent updated (line 2). Otherwise, the next neighborhood is considered (line 3).

3.1 Variable neighborhood descent (VND)

The *Variable neighborhood descent* (VND) method is obtained if the change of neighborhoods is performed in a deterministic way. Its steps are presented in Algorithm 5. In the descriptions of all algorithms that follow we assume that an initial solution x is given. Most local search

Function `VND` (x, k'_{max});
repeat
$\quad k \leftarrow 1$;
$\quad$**repeat**
$\quad\quad x' \leftarrow arg\min_{y \in \mathcal{N}'_k(x)} f(x)$ /* Find the best neighbor in $\mathcal{N}_k(x)$ */ ;
$\quad\quad$`NeighborhoodChange` (x, x', k) /* Change neighborhood */ ;
$\quad$**until** $k = k'_{max}$;
until *no improvement is obtained* ;

Algorithm 5: Steps of the basic VND

heuristics use in their descents a single or sometimes two neighborhoods ($k'_{max} \leq 2$). Note that the final solution should be a local minimum w.r.t. all k'_{max} neighborhoods, and thus chances to reach a global one are larger when using VND than with a single neighborhood structure. Beside this *sequential* order of neighborhood structures in VND above, one can develop a *nested* strategy. Assume e.g. that $k'_{max} = 3$; then a possible nested strategy is: perform VND from Figure 1 for the first two neighborhoods, in each point x' that belongs to the third ($x' \in \mathcal{N}_3(x)$). Such an approach is applied e.g. in Brimberg at al. [2000], Hansen and Mladenović [2001b].

3.2 Reduced VNS

The *Reduced* VNS (RVNS) method is obtained if random points are selected from $\mathcal{N}_k(x)$ and no descent is made. Rather, the values of these new points are compared with that of the incumbent and updating takes place in case of improvement. We assume that a stopping condition has been chosen, among various possibilities, e.g., the maximum CPU time allowed t_{max}, or maximum number of iterations between two improvements. To simplify the description of the algorithms we always use t_{max} below. Therefore, RVNS uses two parameters: t_{max} and k_{max}. Its steps are presented in Algorithm 6. With the function `Shake` represented in line 4, we generate a point x' at random from the k^{th} neighborhood of x, i.e., $x' \in \mathcal{N}_k(x)$.

```
  Function RVNS (x, k_max, t_max) ;
1 repeat
2    k ← 1;
3    repeat
4       x' ← Shake(x, k);
5       NeighborhoodChange (x, x', k) ;
     until k = k_max ;
6    t ← CpuTime()
  until t > t_max ;
```

Algorithm 6: Steps of the Reduced VNS

RVNS is useful for very large instances for which local search is costly. It is observed that the best value for the parameter k_{max} is often 2. In addition, the maximum number of iterations between two improvements is usually used as stopping condition. RVNS is akin to a Monte-Carlo method, but more systematic (see e.g., Mladenović et al. [2003] where results obtained by RVNS were 30% better than those of the Monte-Carlo method in solving a continuous min-max problem). When applied to the p-Median problem, RVNS gave equally good solutions as the *Fast Interchange* heuristic of Whitaker [1983] in 20 to 40 times less time [Hansen et al., 2001].

3.3 Basic VNS

The *Basic VNS* (BVNS) method [Mladenović and Hansen, 1997] combines deterministic and stochastic changes of neighborhood. Its steps are given in Algorithm 7.

```
  Function VNS (x, k_max, t_max);
1 repeat
2    k ← 1;
3    repeat
4       x' ← Shake(x, k)              /* Shaking */;
5       x'' ← FirstImprovement(x')    /* Local search */;
6       NeighborhoodChange(x, x'', k) /* Change neighborhood */;
     until k = k_max ;
7    t ← CpuTime()
  until t > t_max ;
```

Algorithm 7: Steps of the basic VNS

Often successive neighborhoods $\mathcal{N}_k$ will be nested. Observe that point x' is generated at random in step 4 in order to avoid cycling, which might occur if any deterministic rule was applied. In step 5 the first improvement local search (Algorithm 3) is usually adopted. However, it can be replaced with best improvement (Algorithm 2).

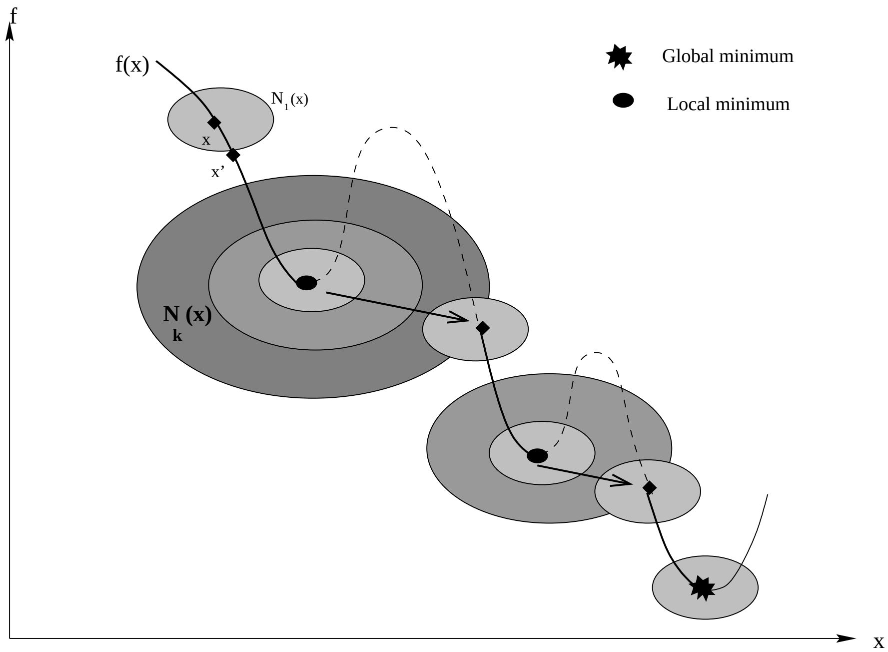

Figure 1: Basic VNS.

3.4 General VNS

Note that the Local search step 5 may be also replaced by VND (Algorithm 5. Using this general VNS (VNS/VND) approach led to the most successful applications reported (see e.g. [Andreatta and Ribeiro, 2002, Brimberg at al., 2000, Canuto et al., 2001, Caporossi and Hansen, 2000, 2004, Caporossi et al., 1999a,b, Hansen and Mladenović, 2001b, Hansen et al., 2006, Ribeiro and de Souza, 2002, Ribeiro et al., 2002]. Steps of the general VNS (GVNS) are given in Algorithm 8 below.

Function `GVNS` $(x, k'_{max}, k_{max}, t_{max})$;
repeat
    $k \leftarrow 1$;
    repeat
        $x' \leftarrow$ `Shake`(x, k);
        $x'' \leftarrow$ `VND`(x', k'_{max}) ;
        `NeighborhoodChange`(x, x'', k);
    until $k = k_{max}$;
    $t \leftarrow$ `CpuTime()`
until $t > t_{max}$;

Algorithm 8: Steps of the general VNS

3.5 Skewed VNS

The skewed VNS (SVNS) method [Hansen et al., 2000] addresses the problem of exploring valleys far from the incumbent solution. Indeed, once the best solution in a large region has been found it is necessary to go quite far to obtain an improved one. Solutions drawn at random in far-away neighborhoods may differ substantially from the incumbent and VNS can then degenerate, to some extent, into the Multistart heuristic (in which descents are made iteratively from solutions generated at random, and which is known not to be very efficient). So some compensation for distance from the incumbent must be made and a scheme called Skewed VNS is proposed for that purpose. Its steps are presented in Algorithms 10 and 11, where the `KeepBest`(x, x') function simply keeps the better between x and x': **if** $f(x') < f(x)$ **then** $x \leftarrow x'$.

Function `NeighborhoodChangeS`(x, x'', k, α);
if $f(x'') - \alpha\rho(x, x'') < f(x)$ **then**
    $x \leftarrow x''$; $k \leftarrow 1$
else
    $k \leftarrow k + 1$
end

Algorithm 9: Steps of Neighborhood change for the Skewed VNS

Function `SVNS` $(x, k_{max}, t_{max}, \alpha)$;
repeat
    $k \leftarrow 1$; $x_{best} \leftarrow x$;
    repeat
        $x' \leftarrow$ `Shake`(x, k) ;
        $x'' \leftarrow$ `FirstImprovement`(x') ;
        `KeepBest` (x_{best}, x);
        `NeighborhoodChangeS`(x, x'', k, α);
    until $k = k_{max}$;
    $x \leftarrow x_{best}$;
    $t \leftarrow$ `CpuTime`$()$;
until $t > t_{max}$;

Algorithm 10: Steps of the Skewed VNS

SVNS makes use of a function $\rho(x, x'')$ to measure distance between the incumbent solution x and the local optimum found x''. The distance used to define the $\mathcal{N}_k$, as in the above examples, could be used also for this purpose. The parameter α must be chosen in order to accept exploring valleys far from x when $f(x'')$ is larger than $f(x)$ but not too much (otherwise one will always leave x). A good value is to be found experimentally in each case. Moreover, in order to avoid frequent moves from x to a close solution one may take a large value for α when $\rho(x, x'')$ is small. More sophisticated choices for a function of $\alpha\rho(x, x'')$ could be made through some learning process.

3.6 Some extensions of Basic VNS

Several easy ways to extend the basic VNS are now discussed. The basic VNS is a descent, first improvement method with randomization. Without much additional effort it could be transformed into a descent-ascent method: in `NeighborhoodChange()` function set also $x \leftarrow x$" with some probability even if the solution is worse than the incumbent (or best solution found so far). It could also be changed into a best improvement method: make a move to the best neighborhood k^* among all k_{max} of them. Its steps are given in Algorithm 12.

Function `BI-VNS` (x, k_{max}, t_{max});
repeat
    $k \leftarrow 1$;
    $x_{best} \leftarrow x$;
    repeat
        $x' \leftarrow$ `Shake`(x, k);
        $x'' \leftarrow$ `FirstImprovement`(x');
        `KeepBest`(x_{best}, x'');
        $k \leftarrow k + 1$;
    until $k = k_{max}$;
    $x \leftarrow x_{best}$;
    $t \leftarrow$ `CpuTime()`
until $t > t_{max}$;

Algorithm 11: Steps of the basic Best Improvement VNS

Another variant of the basic VNS could be to find a solution x' in Step 2a as the best among b (a parameter) randomly generated solutions from the k^{th} neighborhood. There are two possible variants of this extension: (i) perform only one local search from the best point among b; (ii) perform all b local searches and then choose the best. We now give an algorithm of a second type suggested by Fleszar and Hindi [2004]. There, the value of parameter b is set to k. In that way no new parameter is introduced (see Algorithm 12).

Function `FH-VNS` (x, k_{max}, t_{max});
repeat
    $k \leftarrow 1$;
    repeat
        for $\ell = 1$ *to* k **do**
            $x' \leftarrow$ `Shake`(x, k) ;
            $x'' \leftarrow$ `FirstImprovement`(x');
            `KeepBest`(x, x'');
        end
        `NeighborhoodChange`(x, x'', k);
    until $k = k_{max}$;
    $t \leftarrow$ `CpuTime()`
until $t > t_{max}$;

Algorithm 12: Steps of the Fleszar-Hindi extension of the basic VNS

It is also possible to introduce k_{min} and k_{step}, two parameters that control the change of neighborhood process: in the previous algorithms instead of $k \leftarrow 1$ set $k \leftarrow k_{min}$ and instead of $k \leftarrow k + 1$ set $k \leftarrow k + k_{step}$. Steps of Jump VNS are given in Algorithms 13 and 14.

3.7 Variable neighborhood decomposition search

While the basic VNS is clearly useful for approximate solution of many combinatorial and global optimization problems, it remains difficult or long to solve very large instances. As often, size of problems considered is limited in practice by the tools available to solve them

Function `J-VNS` $(x, k_{min}, k_{step}, k_{max}, t_{max})$;
repeat
  $k \leftarrow k_{min}$;
  repeat
    $x' \leftarrow$ `Shake`(x, k);
    $x'' \leftarrow$ `FirstImprovement`(x') ;
    `NeighborhoodChangeJ`$(x, x'', k, k_{min}, k_{step})$;
  until $k = k_{max}$;
  $t \leftarrow$ `CpuTime()`
until $t > t_{max}$;

Algorithm 13: Steps of the Jump VNS

Function `NeighborhoodChangeJ` $(x, x', k, k_{min}, k_{step})$;
if $f(x') < f(x)$ **then**
  $x \leftarrow x'$; $k \leftarrow k_{min}$;
else
  $k \leftarrow k + k_{step}$;
end

Algorithm 14: Neighborhood change or Move or not function

more than by the needs of potential users of these tools. Hence, improvements appear to be highly desirable. Moreover, when heuristics are applied to really large instances their strengths and weaknesses become clearly apparent. Three improvements of the basic VNS for solving large instances are now considered.

The Variable Neighborhood Decomposition Search (VNDS) method [Hansen et al., 2001] extends the basic VNS into a two-level VNS scheme based upon decomposition of the problem. Its steps are presented on Algorithm 15, where t_d is an additional parameter and represents running time given for solving decomposed (smaller sized) problems by VNS.

Function `VNDS` $(x, k_{max}, t_{max}, t_d)$;
1 **repeat**
2   $k \leftarrow 2$;
3   **repeat**
4     $x' \leftarrow$ `Shake` (x, k); $y \leftarrow x' \setminus x$;
5     $y' \leftarrow$ `VNS`(y, k, t_d); $x'' = (x' \setminus y) \cup y'$;
6     $x''' \leftarrow$ `FirstImprovement`(x'');
7   `NeighborhoodChange`(x, x''', k);
  until $k = k_{max}$;
until $t > t_{max}$;

Algorithm 15: Steps of VNDS

For ease of presentation, but without loss of generality, we assumed that the solution x represents the set of some elements. In step 4 we denote with y a set of k solution attributes present in x' but not in x ($y = x' \setminus x$). In step 5 we find the local optimum y' in the space of y; then we denote with x'' the corresponding solution in the whole space S ($x'' = (x' \setminus y) \cup y'$).

We noticed that exploiting some *boundary effects* in a new solution can significantly improve the solution quality. That is why, in step 6, we find the local optimum x''' in the whole space S using x'' as an initial solution. If this is time consuming, then at least a few local search iterations should be performed.

VNDS can be viewed as embedding the classical successive approximation scheme (which has been used in combinatorial optimization at least since the sixties, see e.g. Griffith and Stewart [1961]) in the VNS framework.

3.8 Parallel VNS

Parallel VNS (PVNS) methods are another extension. Several ways for parallelizing VNS have recently been proposed [García-López et al., 2002, Crainic et al., 2004] in solving the p-Median problem. In García-López et al. [2002] three of them are tested : (i) parallelize local search; (ii) augment the number of solutions drawn from the current neighborhood and do local search in parallel from each of them and (iii) do the same as (ii) but updating the information about the best solution found. The second version gave the best results. It is shown in Crainic et al. [2004] that assigning different neighborhoods to each processor and interrupting their work as soon as an improved solution is found gives very good results: best known solutions have been found on several large instances taken from TSP-LIB Reinelt [1991]. Three Parallel VNS strategies are also suggested for solving the Travelling purchaser problem in Ochi et al. [2001].

3.9 Primal-Dual VNS

For most modern heuristics the difference in value between the optimal solution and the obtained one is completely unknown. Guaranteed performance of the primal heuristic may be determined if a lower bound on the objective function value is known. To that end the standard approach is to relax the integrality condition on the primal variables, based on a mathematical programming formulation of the problem. However, when the dimension of the problem is large, even the relaxed problem may be impossible to solve exactly by standard commercial solvers. Therefore, it looks as a good idea to solve dual relaxed problems heuristically as well. In that way we get guaranteed bounds on the primal heuristics performance. The next problem arises if we want to get exact solution within a Branch and bound framework since having the approximate value of the relaxed dual does not allow us to branch in an easy way, e.g., exploiting complementary slackness conditions. Thus, the exact value of the dual is necessary.

In Primal-dual VNS (PD-VNS) [Hansen et al., 2007] we propose one possible general way to get both the guaranteed bounds and the exact solution. Its steps are given in Algorithm 16.

Function `PD-VNS` $(x, k'_{max}, k_{max}, t_{max})$;
`BVNS` $(x, k'_{max}, k_{max}, t_{max})$ /* Solve primal by VNS */;
`DualFeasible`(x, y) /* Find (infeasible) dual such that $f_P = f_D$ */ ;
`DualVNS`(y) /* Use VNS do decrease infeasibility */;
`DualExact`(y) /* Find exact (relaxed) dual */;
`BandB`(x, y) /* Apply branch-and-bound method */ ;

Algorithm 16: Steps of the basic PD-VNS

In the first stage a heuristic procedure based on VNS is used to obtain a near optimal solution. In Hansen et al. [2007] we show that VNS with decomposition is a very powerful technique for large-scale Simple plant location problems (SPLP) up to 15000 facilities × 15000 users. In the second phase, our approach is designed to find an exact solution of the relaxed dual problem. For solving SPLP, this is accomplished in three stages: (i) find an initial dual solution (generally infeasible) using the primal heuristic solution and complementary slackness conditions; (ii) improve the solution by applying VNS on the unconstrained nonlinear form of the dual; (iii) finally, solve the dual exactly using a customized "sliding simplex" algorithm that applies "windows" on the dual variables to reduce substantially the size of the problem. In all problems tested, including instances much larger than previously reported in the literature, our procedure was able to find the exact dual solution in reasonable computing time. In the third and final phase armed with tight upper and lower bounds, obtained respectively, from the heuristic primal solution in phase one and the exact dual solution in phase two, we apply a standard branch-and-bound algorithm to find an optimal solution of the original problem. The lower bounds are updated with the dual sliding simplex method and the upper bounds whenever new integer solutions are obtained at the nodes of the branching tree. In this way we were able to solve exactly problem instances with up to $7,000 \times 7,000$ for uniform fixed costs and 15,000 × 15,000 otherwise.

3.10 Variable neighborhood formulation space search

Traditional ways to tackle an optimization problem consider a given formulation and search in some way through its feasible set $\mathcal{S}$. The fact that a same problem may often be formulated in different ways allows to extend search paradigms to include jumps from one formulation to another. Each formulation should lend itself to some traditional search method, its 'local search' that works totally within this formulation, and yields a final solution when started from some initial solution. Any solution found in one formulation should easily be translatable to its equivalent formulation in any other formulation. We may then move from one formulation to another using the solution resulting from the former's local search as initial solution for the latter's local search. Such a strategy will of course only be useful in case local searches in different formulations behave differently.

This idea was recently investigated in Mladenović et al. [2005] using an approach that systematically changes formulations for solving circle packing problems (CPP). It is shown there that a stationary point of a non-linear programming formulation of CPP in Cartesian coordinates is not necessarily also a stationary point in a polar coordinate system. The method *Reformulation descent* (RD) that alternates between these two formulations until the final solution is stationary with respect to both is suggested. Results obtained were comparable with the best known values, but they were achieved some 150 times faster than by an alternative single formulation approach. In that same paper the idea suggested above of *Formulation space search* (FSS) is also introduced, using more than two formulations. Some research in that direction has been reported in Mladenović [2005], Plastria et al. [2005], Hertz et al. [2008]. One algorithm that uses the variable neighborhood idea in searching through the formulation space is given in Algorithms 17 and 18.

In Figure 2 we consider the CPP case with $n = 50$. The set consists of all mixed formulations, in which some circle centers are given in Cartesian coordinates while the others are given in polar coordinates. Distance between two formulations is then the number of centers whose coordinates are expressed in different systems in each formulation. Our FSS starts with the RD solution i.e., with $r_{curr} = 0.121858$. The values of k_{min} and k_{step} are set to 3 and the value of k_{max} is set to $n = 50$. We did not get improvement with $k_{curr} = 3, 6$ and 9. The

Function `FormulationChange`$(x, x', \phi, \phi', \ell)$;
if $f(\phi', x') < f(\phi, x)$ **then**
  $\phi \leftarrow \phi'$; $x \leftarrow x'$; $\ell \leftarrow \ell_{min}$
else
  $\ell \leftarrow \ell + \ell_{step}$;
end

Algorithm 17: Formulation change function

Function `VNFSS`(x, ϕ, ℓ_{max});
repeat
  $\ell \leftarrow 1$ /* Initialize formulation in $\mathcal{F}$ */;
  while $\ell \leq \ell_{max}$ **do**
    `ShakeFormulation`(x,x',ϕ,ϕ',ℓ) /* Take $(\phi',x') \in (N_\ell(\phi),\mathcal{N}(x))$ at random */;
    `FormulationChange`(x,x',ϕ,ϕ',ℓ) /* Change formulation */;
  end
until *some stopping condition is met* ;

Algorithm 18: Reduced variable neighborhood FSS

next improvement was obtained for $k_{curr} = 12$. This means that a “mixed” formulation with 12 polar and 38 Cartesian coordinates is used. Then we turn again to the formulation with 3 randomly chosen circle centers, which was unsuccessfull, but obtained a better solution with 6, etc. After 11 improvements we ended up with a solution with radius $r_{max} = 0.125798$.

4 Applications

Applications of VNS, or of hybrids of VNS combined with other metaheuristics are diverse and numerous. We next review them.

4.1 Industrial applications

Considering first industrial applications, the oil industry provided many problems. These include the design of an offshore pipeline network [Brimberg et al., 2003], the pooling problem [Audet et al., 2004] and scheduling of walkover rigs for Petrobras [Aloise et al., 2006].

4.2 Design problems in communication

Design problems in communications approached with VNS include cable layout [Costa et al., 2002], spread spectrum radar polyphase code design [Mladenović et al., 2003], topological design of a Yottabit-per-second lattice network [Degila and Sansò, 2004], distribution networks [Lapierre et al., 2004], the ring star Network [Dias et al., 2006], ATM networks [Loudni et al., 2006], SDH-WDM networks [Melián, 2006, Melián et al., 2008, Höller et al., 2008], SAW filters [Tagawa et al., 2007] and other optimization problems in computer communications [Ribeiro et al., 2007].

Costa et al. [2002] apply variable neighborhood decomposition search for the optimization of power plant cable layout. Mladenović et al. [2003] use VNS for solving a spread spectrum

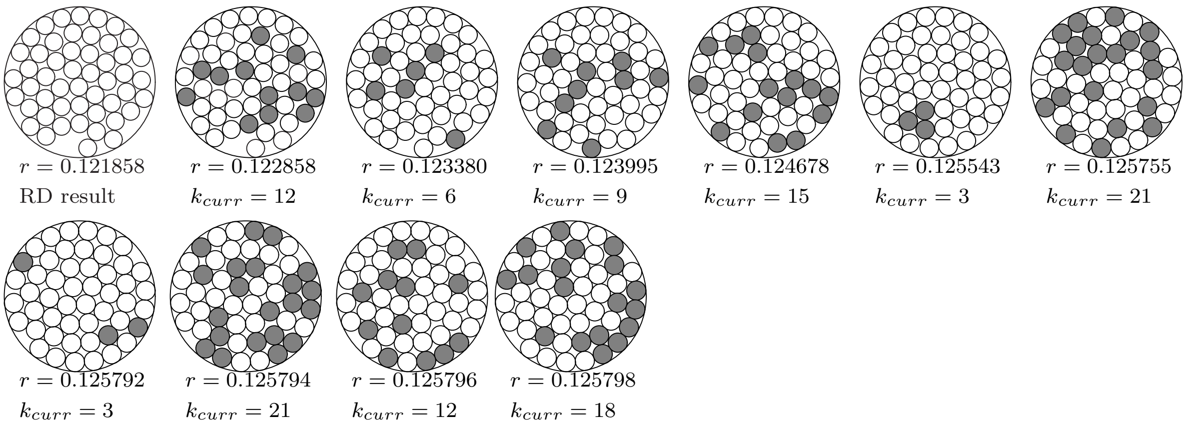

Figure 2: Reduced FSS for PCC problem and $n = 50$.

radar polyphase code design problem. Degila and Sansò [2004] propose a VNS to deal with the topological design of a yotta-bit-per-second (1 yotta = 10^{24}) multidimensional network based upon agile optical cores that provides fully meshed connectivity with direct optical paths between edge nodes that are electronically controlled. Lapierre et al. [2004] consider the application of a TS-VNS hybrid for designing distribution networks with transshipment centers. Dias et al. [2006] use a General Variable Neighborhood Search (GVNS) to improve the quality of the solution obtained with a Greedy Randomized Adaptive Search Procedure (GRASP) for the ring star problem. In Loudni et al. [2006] a difficult real-life network problem of France Telecom R&D, the on-line resources allocation for ATM networks with rerouting, is solved by VNS/LDS+CP.

In Melián [2006], Höller et al. [2008], Melián et al. [2008] the design of SDH/WDM networks is tackled with VNS that is compared with other methods. The application of VNS in the design of SDH/WDM networks is proposed in Melián et al. [2008], it is improved with the use of adaptive memory mechanism in [Melián, 2006] and by applying a pilot method in [Höller et al., 2008]. Tagawa et al. [2007] deal with the robust design of Surface Acoustic Wave (SAW) filters. Ribeiro et al. [2007] consider VNS and other metaheuristics for optimization problems in computer communications. [Meric et al., 2004] apply VNS for optical routing in networks using latin routers.

4.3 Location problems

Location problems have also attracted much attention of the VNS researchers and practitioners. Among discrete models the p-median has been the most studied [Brimberg and Mladenović, 1996, Hansen and Mladenović, 1997, Hansen et al., 2001, García-López et al., 2002, Mladenović et al., 2007, Hansen and Mladenović, 2008]. Hansen et al. [2001] introduces variable neighborhood decomposition search solving the p-median problem. Hansen and Mladenović [2008] complete the comparative analysis in Alba and Domínguez [2006] by detailed comparison of several versions of VNS with other metaheuristics for the p-median problem.

Other discrete location problems solved with VNS are the p-center problems [Mladenović et al., 2003], the maximum capture problem [Benati and Hansen, 2002] and several variants of the p-median problem [Domínguez-Marín et al., 2005, Fathali and Kakhki, 2006, Fleszar and Hindi, 2008, Pérez et al., 2007, Osman and Ahmadi, 2007]. Domínguez-Marín et al.

[2005] deal with solving the Discrete Ordered Median Problem, Fathali and Kakhki [2006] apply VNS to the p-median problem with pos/neg weights, Fleszar and Hindi [2008] solve the capacitated p-median problem and Pérez et al. [2007] propose a hybrid that combines VNS with Path Relinking for the p-hub median problem. Osman and Ahmadi [2007] investigate different search and selection strategies including the Variable neighborhood Search descent for the capacitated p-median problem with single source constraint. Moreno-Pérez et al. [2003] propose a variable neighborhood tabu search hybrid and consider its application to the median cycle problem.

Among continuous models, the multi-source Weber problem is first addressed in Brimberg at al. [2000] and in Brimberg et al. [2004] with constant opening costs. Brimberg et al. [2006] use VNS in a decomposition strategies for large-scale instances. Other mixed location problems that appeared in the VNS literature are found in [Brimberg et al., 2008, Ljubic, 2007, Hansen et al., 2007]. Brimberg et al. [2008] apply VNS to the maximum return-on-investment plant location problem with market share. Ljubic [2007] propose a hybrid VNS for a the Connected Facility Location problem that combines the facility location problem and the Steiner Tree Problem in graphs. Hansen et al. [2007] apply a Primal-Dual Variable Neighborhood Search for the Simple Plant Location Problem. Finally, Bischoff and Dächert [2008] use VNS and other heuristics for a generalized class of continuous location–allocation problems and Jabalameli and Ghaderi [2008] hybrid algorithms that combine GA and VNS for the uncapacitated continuous location–allocation problem.

The use of VNS to solve the quadratic assignment problems is discussed in Drezner et al. [2005], Zhang et al. [2005], Han et al. [2007] and Liu and Abraham [2007]. Drezner et al. [2005] analyse the difficulty in QAP instances for metaheuristic approaches and Zhang et al. [2005] use a variable neighborhood search with permutation distance. Han et al. [2007] use a VNS-ANT hybrid and Liu and Abraham [2007] a fuzzy VNS-PSO hybrid algorithm. Yang et al. [2007] apply optimization strategies based on Simulated Annealing and Variable neighborhood Search for the base station location problem in a WCDMA (wideband code-division multiple access) network. Pacheco et al. [2008] use VNS to solve the classical Maximum Covering Location Problem for locating health resources. Wollenweber [2008] use several hybrids with VNS for a multi-stage facility location problem with staircase costs and splitting of commodities.

4.4 Data mining

VNS proved to be a very efficient tool in cluster analysis. In particular, the J-Means heuristic combined with VNS appears to be the state-of-the-art for heuristic solution of minumum sum-of-square clustering [Hansen and Mladenović, 2001b, Belacel et al., 2002, 2004]. Combined with stabilized column generation [du Merle et al., 1999] it leads to the presently most efficient exact algorithm for that problem [du Merle et al., 2000]. Such an approach has also been applied by [Hansen and Perron, 2007] to solve the $\mathcal{L}_1$ embeddebility problem for data sets.

Other clustering problems appear in [Belacel et al., 2004, Negreiros and Palhano, 2006, Brusco and Steinley, 2007a,b, Benati, 2008, Brusco et al., 2008]. Belacel et al. [2004] use Variable Neighborhood Search Metaheuristic for Fuzzy Clustering cDNA Microarray Gene Expression Data. Negreiros and Palhano [2006] propose a constructive procedure followed by a Variable neighborhood Search to solve the capacitated centred clustering problem. Brusco and Steinley [2007a] compare a variable neighborhood search method with the classical k-means for the clustering of two-mode proximity binary matrices. and Brusco and Steinley [2007b] compare heuristic procedures for Minimum Within-Cluster Sums of Squares Partitioning. Benati [2008] applies VNS to categorical data fuzzy clustering.

An other important Data Mining task that has been managed with VNS is classification [Pacheco et al., 2007, Karam et al., 2007, Hansen et al., 2007, Belacel et al., 2007, Carrizosa et al., 2007]. Pacheco et al. [2007] use of VNS in variable selection and determination of the linear discrimination function coefficients. Karam et al. [2007] perform arbitrary-norm hyperplane separation by Variable neighborhood Search. Hansen et al. [2007] apply VNS for colour image quantization. Belacel et al. [2007] propose a VNS heuristic for learning the parameters of the multiple criteria classification method PROAFTN from data. Carrizosa et al. [2007] use VNS for the selection of the Globally Optimal Prototype Subset for Nearest-Neighbor Classification.

4.5 Graph problems

In addition to some design problems in communications and most of the location problems, other combinatorial optimization problems on graphs to which VNS has been applied include the max-cut problem [Festa et al., 2002], the median cycle problem [Moreno-Pérez et al., 2003], the clique problem [Hansen et al., 2004], the vertex coloring problem [Avanthay et al., 2003, Hertz et al., 2008] and the k-Cardinality Subgraph Problem [Brimberg et al., 2008]. A VNS is proposed for the max-cut problem in a graph and compared with other metaheuristics in Festa et al. [2002] and an hybridization between a memetic algorithm and VNS is proposed for the same problem by [Duarte et al. , 2005]. Moreno-Pérez et al. [2003] propose a Variable neighborhood Tabu Search hybrid for the median cycle problem. Hansen et al. [2004] propose and test a basic Variable neighborhood Search that combines greedy with the simplicial vertex test in its descent step for maximum clique problem. For the graph coloring problem, Avanthay et al. [2003] propose an adaptation of the VNS metaheuristic, [Galinier and Hertz, 2006] present a survey of local search methods and Hertz et al. [2008] analyze the variable space search methodology that extends the Formulation Space Search (FSS). Brimberg et al. [2008] propose a new heuristic based on Variable neighborhood Search for the k-Cardinality Subgraph Problem in contrast with the constructive heuristics proposed in the literature.

Graph problems involving trees tackled with VNS include the Steiner problem [Ribeiro et al., 2002], the prize-collecting Steiner tree problem [Canuto et al., 2001], the k-cardinality tree [Mladenović and Urošević, 2003, Urošević et al., 2004, Brimberg et al., 2006], the degree constrained spanning tree problem [Ribeiro and de Souza, 2002, de Souza and Martins, 2008], the generalized minimum spanning tree problem [Hu et al., 2008] and the minimum labelling spanning tree problem Consoli et al. [2008]. VNS is used in Canuto et al. [2001] as a post-optimization procedure for a multistart local search algorithm for the prize-collecting Steiner tree problem, based on the generation of initial solutions by a primal-dual algorithm using perturbed node prizes. Ribeiro et al. [2002] uses a hybrid VNS-GRASP with perturbations for the Steiner problem in graphs. Mladenović and Urošević [2003] propose to use a VNS for the edge weighted k-cardinality tree problem Urošević et al. [2004] propose a variable neighborhood decomposition search for the same problem and Brimberg et al. [2006] for the vertex weighted k-cardinality tree problem. Ribeiro and de Souza [2002] propose the use of VNS for degree constrained minimum spanning tree problem and de Souza and Martins [2008] use a Skewed VNS enclosing second order algorithm for the same problem. Hu et al. [2008] propose a VGNS approach that uses three different neighborhood types to solve the generalized minimum spanning tree problem. Finally, a VNS is used to solve the minimum labelling spanning tree problem by Consoli et al. [2008].

4.6 Knapsack and packing problems

Another important class of problems solved with VNS and its variants and hybrids are the knapsack and packing problems. In Puchinger et al. [2006] a relaxation guided variable neighborhood search is applied to the Multidimensional Knapsack Problem and to its core problems. The paper Puchinger and Raidl [2008] constitutes an excellent illustration of a dynamic ordering of the neighborhood structures embedded in a Variable Neighborhood Descent algorithm that is used to solve also the Multidimensional Knapsack Problem. VNS has also been successfully applied to the bin packing problem in [Fleszar and Hindi, 2002] and to the Strip Packing Problem in [Beltrán et al., 2004].

Circle and sphere packing have also been approached with VNS [Mladenović et al., 2005, 2007, Kucherenko et al., 2007]. Mladenović et al. [2005] introduce the reformulation descent that is applied to circle packing problems and Mladenović et al. [2007] the Formulation Space Search for the same Problems. Kucherenko et al. [2007] use VNS to solve the Kissing Number Problem; i.e the problem of determining the maximum number of D-dimensional spheres of radius r that can be adjacent to a central sphere of radius r.

4.7 Mixed integer problems

Heuristics may help to find a feasible solution or an improved and possibly optimal solution to large and difficult mixed integer programs. The local branching method of Fischetti and Lodi [2003] does that, in the spirit of VNS. For further developments see Fischetti et al. [2004] and Hansen et al. [2006]. Gutjahr et al. [2007] uses VNS approach for Noisy Problems and its application to Project Portfolio Analysis.

4.8 Time tabling

Timetabling and related manpower organization problems can be well solved with VNS. They include the exam proximity problem [Cote et al., 2005], the design of balanced MBA student teams [Desrosiers et al., 2005], apportioning the European Parliament [Villa et al., 2006], orienteering problems [Sevkli and Sevilgen, 2006, Archetti et al., 2007], detailed layout planning for irregularly-shaped machines [Bock and Hoberg, 2007] and nurse rostering [Burke et al., 2004, 2008]. Cote et al. [2005] use a simplified Variable Neighborhood Descent in a hybrid multi-objective evolutionary algorithm for the uncapacitated exam proximity problem. Sevkli and Sevilgen [2006] propose a Variable Neighborhood Search approach for the Orienteering Problem and Archetti et al. [2007] propose VNS to solve the Team Orienteering Problem (TOP) that is the generalization to the case of multiple tours of the Orienteering Problem, known also as Selective Traveling Salesman Problem.

4.9 Scheduling

Last years also several scheduling problem have been efficiently solved with VNS approach. They include single machine [Gupta and Smith, 2006, Lin and Ying, 2008, Liao and Cheng, 2007] and parallel machines [Anghinolfi and Paolucci, 2007, De Paula et al., 2007], multi-objective scheduling [Gagné et al., 2005, Qian et al., 2006], job shop scheduling [Sevkli and Aydin, 2006a,b, 2007, Gao et al., 2008], flow shop [Blazewicz et al., 2005, 2008, Zobolas et al., 2008], resource-constrained project scheduling [Fleszar and Hindi, 2004, Kolisch and Hartmann, 2007] and other scheduling problems [Pan et al., 2007b, Almada-Lobo et al., 2008, Dahal et al., 2008].

4.9.1 Single Machine Scheduling

Gupta and Smith [2006] use a VNS algorithm for single machine total tardiness scheduling with sequence dependent setups. Lin and Ying [2008] proposed a hybrid Tabu-VNS meta-heuristic approach for single-machine tardiness problems with sequence-dependent setup times. Liao and Cheng [2007] propose a variable neighborhood search for minimizing single machine weighted earliness and tardiness with common due date.

4.9.2 Parallel Machine Scheduling

Anghinolfi and Paolucci [2007] propose a hybrid metaheuristic approach which integrates several features from tabu search, simulated annealing and Variable neighborhood Search for a Parallel machine total tardiness scheduling problem. De Paula et al. [2007] apply VNS for solving parallel machines scheduling problems with sequence-dependent.

4.9.3 Multiobjective Scheduling

Gagné et al. [2005] use compromise programming with tabu-VNS metaheuristic for the solution of multiple-objective scheduling problems. Qian et al. [2006] deal with multi-objective flow shop scheduling using differential evolution.

4.9.4 Job shop Scheduling

Sevkli and Aydin [2006a] and Sevkli and Aydin [2006b] use Variable neighborhood Search algorithms for job shop scheduling problems. Sevkli and Aydin [2007] propose parallel Variable neighborhood Search algorithms and Gao et al. [2008] propose a hybrid GA/VND and Pan et al. [2007b] a PSO/VNS hybrid heuristics for these problems. Liu et al. [2006] propose a variable neighborhood particle swarm optimization for multi-objective flexible job-shop scheduling problems.

4.9.5 Flow shop Scheduling

Blazewicz et al. [2005] uses VNS for late work minimization in two-machine flow shop with common due date. In Pan et al. [2007a] VNS and other three Metaheuristic approaches are proposed for a no-wait flow shop problem and in Blazewicz et al. [2008] VNS and other two Metaheuristic for the two-machine flow-shop problem with weighted late work criterion and common due date. Zobolas et al. [2008] design a GA/VNS hybrid to minimize makespan in permutation flow shop scheduling problems. In Tasgetiren et al. [2004] a simple but very efficient local search based on the VNS is embedded in the PSO algorithm to solve the for Permutation Flow shop Sequencing Problem. Liao et al. [2007] apply VNS for flow shop scheduling problems and Tasgetiren et al. [2007] consider the makespan and total flow time minimization in the permutation flow shop sequencing problem.

4.9.6 Resource-Constrained Project Scheduling

Fleszar and Hindi [2004] propose to solve the resource-constrained project scheduling problem by a variable neighborhood search and Kolisch and Hartmann [2007] include VNS in an experimental investigation of heuristics for resource-constrained project scheduling. Bouffard and Ferland [2007] improve simulated annealing with variable neighborhood search to solve the resource-constrained scheduling problem.

4.9.7 Car Sequencing

Several papers deal with car sequencing [Prandtstetter and Raidl, 2008, Gavranović, 2008, Ribeiro et al., 2008a,b, Joly and Frein, 2007] with VNS. Prandtstetter and Raidl [2008] use a hybrid variable neighborhood search for the car sequencing problem and Gavranović [2008] apply VNS to car-sequencing problem with colors. Ribeiro et al. [2008a] propose a set of heuristics based on the paradigms of the VNS and ILS metaheuristics for a multi-objective real-life car sequencing problem with painting and assembly line constraints and Ribeiro et al. [2008b] provide an efficient implementation of the VNS/ILS heuristic for this real-life car sequencing problem. Joly and Frein [2007] use VNS to tackle an industrial car sequencing problem considering paint and assembly shop objectives

4.9.8 Other scheduling problems

Davidović et al. [2005] use a variable neighborhood search heuristics for multiprocessor scheduling with communication delays. Higgins et al. [2006] apply VNS to the scheduling of brand production and shipping within a sugar supply chain and Lejeune [2006] also consider supply chain planning. Liang and Chen [2007] tackle redundancy allocation of series-parallel systems using a variable neighborhood search algorithm.

Lusa and Potts [2008] use a Variable neighborhood Search algorithm for the constrained task allocation problem Liang et al. [2007] apply Variable neighborhood Search for redundancy allocation problems. Remde et al. [2007] use reduced Variable neighborhood Search and hyperheuristic approaches to tackle subproblems in an Exact/Hybrid heuristic for Workforce Scheduling. Xhafa [2007] considers a hybrid evolutionary metaheuristic based on memetic algorithms and Variable Neighborhood Search for job scheduling on computational grids.

Almada-Lobo et al. [2008] report the use of a VNS approach to production planning and scheduling in the glass container industry. Dahal et al. [2008] apply a constructive search and VNS to tackle a complex real world workforce scheduling problem.

4.10 Vehicle Routing Problems

4.10.1 TSP and extensions

VNS is used for the TSP [Hansen and Mladenović, 1999b, 2006, Burke et al., 2001b, Carrabs et al., 2007, Hu and Raidl, 2008]. Hansen and Mladenović [1999b, 2006] consider basic VNS for the euclidean TSP. Burke et al. [2001b] apply guided variable neighborhood search methods for the asymmetric travelling salesman problem. VNS for the Pickup and Delivery Travelling Salesman Problem is considered in Carrabs et al. [2007]. Hu and Raidl [2008] study the effectiveness of neighborhood structures within a VNS approach for the Generalized Traveling Salesman Problem.

4.10.2 VRP and extensions

Standard versions of vehicle routing problems were solved by VNS or hybrids [Crispim and Brandao, 2001, Rousseau et al., 2002, Bräysy, 2003, Polacek et al., 2004, Melechovsky et al., 2005, Repoussis et al., 2006, Irnich et al., 2006, Kytöjoki et al., 2007, Fleszar et al., 2008, Geiger and Wenger, 2007]. A variable neighborhood descent is applied to the vehicle routing problem with backhauls in Crispim and Brandao [2001]. Rousseau et al. [2002] use a variable neighborhood descent to take advantage of different neighborhood structures for the vehicle routing problem. An interesting development of reactive variable neighborhood search for the vehicle routing problem with time windows appears in [Bräysy, 2003] Polacek et al. [2004] uses a variable neighborhood search for the multi depot vehicle routing problem with time windows. A hybrid metaheuristic merging Variable Neighborhood Search and Tabu Search to the location-routing problem with non-linear costs in [Melechovsky et al., 2005]. Repoussis et al. [2006] propose aa reactive greedy randomized variable neighborhood Tabu search for the vehicle routing problem with time windows. Irnich et al. [2006] introduces sequential search as a generic technique for the efficient exploration of local-search neighborhoods such as VNS and consider its application to vehicle-routing problems. Kytöjoki et al. [2007]propose an efficient variable neighborhood search heuristic for very large scale vehicle routing problems. Fleszar et al. [2008] propose an effective VNS for for the open vehicle routing problem. Geiger and Wenger [2007] use VNS within a interactive resolution method for multi-objective vehicle routing problems

4.10.3 Practical applications

VNS has also been useful for practical applications of routing problems. Cowling and Keuthen [2005] examine iterated approaches of Large-Step Markov Chain and Variable neighborhood Search type and investigate their performance when used in combination with a embedded search heuristic for routing optimization. An VNS based on-line method is proposed and tested in Goel and Gruhn [2008] for the General Vehicle Routing Problem. The solution methodology proposed by Repoussis et al. [2007] hybridizes in a reactive fashion systematic diversification mechanisms of Greedy Randomized Adaptive Search Procedures with Variable Neighborhood Search for intensification local search for a Real Life Vehicle Routing Problem.

4.10.4 Arc routing and waste collection

Variable Neighborhood Search has also been applied for arc routing problems [Hertz and Mittaz, 2001, Ghiani et al., 2002, Polacek et al., 2008]. Hertz and Mittaz [2001] use a variable neighborhood descent algorithm for the undirected capacitated arc routing problem. Polacek et al. [2008] develop a basic variable neighborhood search algorithm to solve the Capacitated Arc Routing Problem with Intermediate Facilities.

VNS has also been applied to waste collection in Nuortio et al. [2006] and Del Pia and Filippi [2006] Nuortio et al. [2006] use VNS in an improved route planning and scheduling of waste collection and transport and Del Pia and Filippi [2006] use a variable neighborhood descent algorithm for a real waste collection problem with mobile depots.

4.10.5 Fleet Sheet Problems

For fleet sheet problems, VNS has been applied in [Yepes and Medina, 2006] and [Paraskevopoulos et al., 2008]. Yepes and Medina [2006] present a three-step local search algorithm based on a probabilistic variable neighborhood search for the vehicle routing problem with a heterogeneous fleet of vehicles and soft time windows. Paraskevopoulos et al. [2008] present a Reactive Variable Neighborhood Tabu Search for the Heterogeneous Fleet Vehicle Routing Problem with time windows.

4.10.6 Extended vehicle routing problems

VNS for extended VRP's has been considered in [Polacek et al., 2007, Zhao et al., 2008, Vogt et al., 2007, Hemmelmayr et al., 2008]. Polacek et al. [2007] use VNS to assign customers to days and determines routes for a travelling salesperson for scheduling periodic customer visits. Zhao et al. [2008] apply a variable large neighborhood search (VLNS) algorithm, which is a special case of VNS for an inventory/routing problem in a three-echelon logistics system. Vogt et al. [2007] present a heuristic for this problem based on variable neighborhood tabu search for the single vehicle routing allocation problem. Hemmelmayr et al. [2008] propose a VNS heuristic for Periodic Routing Problems.

4.11 Problem in Biosciences and Chemistry

Andreatta and Ribeiro [2002] propose VNS heuristics for the phylogeny problem and Ribeiro and Vianna [2005] use a GRASP/VND heuristic for this problem with a new neighborhood structure. Kawashimo et al. [2006] apply VNS to DNA Sequence Design and Liberti et al. [2008] propose a double VNS with smoothing for the molecular distance geometry problem. Santana et al. [2008] illustrate another example of hybridization of metaheuristics through the combination of VNS and Estimation Distribution Algorithms (EDAs); they present the first attempt to combine these two methods testing it on the protein side chain placement problem. Belacel et al. [2004] use VNS for Fuzzy Clustering of cDNA microarray gene expression data and Dražić et al. [2008] use a continuous VNS heuristic for finding the three-dimensional structure of a molecule.

A Multi-Start VNS hybrid (MSVNS) is applied to the protein structure comparison problem in [Pelta et al., 2008]; this is the first time a metaheuristic is applied to this important problem of bio-informatics area. The Maximum Contact Map Overlap (Max-CMO) model of protein structure comparison models the proteins as a graph of the contacts between the protein residues to perform the comparison. The proposed MSVNS method is currently the best heuristic algorithm for the Max-CMO model both in terms of optimization and in terms of the biological relevance of its results. The method is biologically relevant since the algorithm proven to be good enough to detect similarities at SCOP's family and CATH's architecture levels.

4.12 Continuous optimization

Mladenović et al. [2008] propose a General VNS for the continuous optimization and Dražić et al. [2006] a VNS-based Software for Global Optimization. Audet et al. [2008] deal with Nonsmooth optimization through Mesh Adaptive Direct Search and VNS. Brimberg et al. [2006] use VNS in a decomposition strategies for large-scale continuous location-allocation problems. Solving the unconstrained optimization problem by VNS has been considered in Toksari and Güner [2007]. Ling et al. [2007] use a modified VNS metaheuristic for max-bisection problems.

4.13 Other optimization problems

Some further optimization problems solved with VNS include study of the dynamics of handwriting [Caporossi et al., 2004], the problem of multi-item, single level, capacitated, dynamic lot-sizing with set-up times [Hindi et al., 2003], the linear ordering problem [García et al., 2006] and the minimum cost berth allocation problem [Hansen et al., 2008].

Mori and Tsunokawa [2005] use a variable neighborhood tabu search for capacitor placement in distribution systems. Haugland [2007] develop a randomized search heuristic, which in some sense resembles VNS, for the Subspace Selection Problem. Hansen and Perron [2007] use VNS and Tabu search for the $\mathcal{L}_1$-embeddebility of real-valued distance matrices. Hemmelmayr et al. [2008] apply a solution approaches based on integer programming and variable neighborhood search to organize the delivery of blood products to Austrian hospitals for the blood bank of the Austrian Red Cross for Eastern Austria.

The (VNS/LDS+CP) procedure combines a VNS scheme with Limited Discrepancy Search (LDS) using Constraint Propagation (CP); Loudni and Boizumault [2008] apply it for solving optimization problems in anytime contexts.

VNS has also been used to satisfiability problems. Hansen et al. [2000] use VNS for weighted maximum satisfiability problem. Ognjanović et al. [2005], Jovanović et al. [2007] and Sevkli and Aydin [2007] use VNS for the probabilistic satisfiability problem. [Hansen and Perron, 2008] use VNS to solve the subproblem in a column generation approach which merges the local and global approaches to probabilistic satisfiability.

4.14 Discovery science

In all these applications VNS is used as an optimization tool. It can also lead to results in "discovery science", i.e., help in the development of theories. This has been done for graph theory in a long series of papers with the common title "Variable neighborhood search for extremal graphs" and reporting on development and applications of the system AutoGraphiX (AGX) [Caporossi and Hansen, 2000, 2004, Aouchiche et al., 2006c]. This system addresses the following problems:

- Find a graph satisfying given constraints;
- Find optimal or near optimal graphs for an invariant subject to constraints;
- Refute a conjecture;
- Suggest a conjecture (or repair or sharpen one);
- Provide a proof (in simple cases) or suggest an idea of proof.

A basic idea is then to consider all of these problems as parametric combinatorial optimization problems on the infinite set of all graphs (or in practice some smaller subset) with a generic heuristic. This is done by applying VNS to find extremal graphs, with given number n of vertices (and possible also a given number of edges). Then a VND with many neighborhoods is used. Those neighborhoods are defined by modifications of the graphs such as removal or addition of an edge, rotation of an edge, and so forth. Once a set of extremal graphs, parameterized by their order, is found their properties are explored with various data mining techniques and lead to conjectures, refutations, and simple proofs or ideas of proof.

The current list of titles of papers in the series "VNS for extremal graphs" is given in Table 1.

Another list of papers, not included in this series is given in Table 2.

Table 1: List of papers in the series "VNS for extremal graphs"

	Author(s)	Title
1.1	Caporossi and Hansen [2000]:	*The AutoGraphiX System.*
1.2	Caporossi et al. [1999a]:	*Finding graphs with extremal energy.*
1.3	Cvetković et al. [2001]:	*On the Largest Eigenvalue of Color-Constrained Trees.*
1.4	Caporossi et al. [1999b]:	*Chemical trees with extremal connectivity index.*
1.5	Caporossi and Hansen [2004]:	*Three ways to automate finding conjectures.*
1.6	Hansen and Mélot [2003]:	*Analysing Bounds for the Connectivity Index.*
1.7	Fowler et al. [2001]:	*Polyenes with maximum HOMO-LUMO gap.*
1.8	Aouchiche et al. [2001]:	*Variations on Graffiti 105.*
1.9	Hansen and Mélot [2005]:	*Bounding the Irregularity of a Graph.*
1.10	Gutman et al. [2005]:	*Comparison of irregularity indices for chemical trees.*
1.11	Belhaiza et al. [2007]:	*Bounds on Algebraic Connectivity.*
1.12	Hansen et al. [2005]:	*A Note on the Variance of Bounded Degrees in Graphs.*
1.13	Aouchiche and Hansen [2005c]:	*À propos de la maille (French).*
1.14	Aouchiche et al. [2006c]:	*The AutoGraphiX 2 System.*
1.15	Hansen and Stevanović [2005]:	*On Bags and Bugs.*
1.16	Aouchiche et al. [2008]:	*Some conjectures related to the largest eigenvalue of a graph.*
1.17	Aouchiche et al. [2005d]:	*Further Conjectures and Results about the Index.*
1.18	Aouchiche et al. [2006a]:	*Conjectures and Results about the Randic Index.*
1.19	Aouchiche et al. [2007b]:	*Further Conjectures and Results about the Randic Index.*
1.20	Aouchiche et al. [2007c]:	*Automated Comparison of Graph Invariants.*
1.21	Aouchiche et al. [2006b]:	*Conjectures and Results About the Independence Number.*
1.22	Aouchiche et al. [2007a]:	*Extending Bounds for Independence to Upper Irredundance.*
1.23	Hansen and Vukicević [2006]:	*On the Randic Index and the Chromatic Number.*
1.24	Sedlar et al. [2007a]:	*Conjectures and Results About the Clique Number.*
1.25	Sedlar et al. [2007b]:	*Products of Connectivity and Distance Measures.*
1.26	Aouchiche et al. [2007]:	*Nouveaux résultats sur la maille (French).*
1.27	Aouchiche, Caporossi and Hansen [2007g]:	*Families of Extremal Graphs.*

Table 2: A further list of papers on AGX

	Author(s)	Title
1	Caporossi et al. [1999a]:	*Trees with Palindromic Hosoya Polynomials.*
2	Gutman et al. [1999]:	*Alkanes with small and large Randić connectivity indices.*
3	Hansen [2002]	*Computers in Graph Theory.*
4	Hansen and Mélot [2002]	*Computers and Discovery in Algebraic Graph Theory.*
5	Caporossi et al. [2003]:	*Graphs with maximum connectivity index.*
6	Hansen [2005]	*How far is, should and could be conjecture-making in graph theory an automated process?*
7	Aouchiche et al. [2005b]:	*AutoGraphiX: A Survey.*
8	Aouchiche and Hansen [2007d]:	*Automated Results and Conjectures on Average Distance in Graphs.*
9	Aouchiche and Hansen [2007e]:	*On a Conjecture about the Randic Index.*
10	Stevanovic et al. [2008]:	*On the Spectral Radius of Graphs with a Given Domination Number.*
11	Aouchiche and Hansen [2008a]:	*Bounding Average Distance Using Minimum Degree.*
12	Aouchiche and Hansen [2008b]:	*Nordhaus-Gaddum Relations for Proximity and Remoteness in Graphs.*

Papers in these two lists cover variety of topics:

(i) Principles of the approach (1.1, 1.5) and its implementation (1.14);

(ii) Applications to spectral graph theory, e.g., bounds on the index for various families of graphs, graphs maximizing the index subject to some conditions (1.3, 1.11, 1.16, 1.17, 2.7);

(iii) Studies of classical graph parameters, e.g., independence, chromatic number, clique number, average distance (1.13, 1.21, 1.22, 1.24, 1.25, 1.26, 2.8);

(iv) Studies of little known or new parameters of graphs, e.g., irregularity, proximity and remoteness (1.9, 2.9)

(v) New families of graphs discovered by AGX, e.g., bags which are obtained from complete graphs by replacing an edge by a path, and bugs which are obtained by cutting the paths of a bag (1.15, 1.27);

(vi) Applications to mathematical chemistry, e.g., study of chemical graph energy, and of the Randić index (1.4, 1.6, 1.7, 1.10, 1.18, 1.19, 2.2, 2.3, 2.6);

(vii) Results of a systematic study of 20 graph invariants, which led to almost 1500 new conjectures, more than half of which were proved by AGX and over 300 by various mathematicians (1.20);

(viii) Refutation or strengthening of conjectures from the literature (1.8, 2.1, 2.6);

(ix) Surveys and discussions about various discovery systems in graph theory, assessment of the state-of-the-art and the forms of interesting conjectures together with proposals for the design of more powerful system (2.4, 2.5).

5 Conclusions

The general schemes of Variable Neighborhood Search have been presented, discussed and illustrated by examples. In order to evaluate the VNS research program, one needs a list of desirable properties of metaheuristics. The following eight ones are presented in Hansen and Mladenović (2003):

(i) *Simplicity:* the metaheuristic should be based on a simple and clear principle, which should be largely applicable;

(ii) *Precision:* steps of the metaheuristic should be formulated in precise mathematical terms, independent from the possible physical or biological analogy which was an initial source of inspiration;

(iii) *Coherence:* all steps of heuristics for particular problems should follow naturally from the metaheuristic's principle;

(iv) *Efficiency:* heuristics for particular problems should provide optimal or near-optimal solutions for all or at least most realistic instances. Preferably, they should find optimal solutions for most problems of benchmarks for which such solutions are known, when available;

(v) *Effectiveness:* heuristics for particular problems should take moderate computing time to provide optimal or near-optimal solutions;

(vi) *Robustness:* performance of heuristics should be consistent over a variety of instances, i.e., not just fine-tuned to some training set and less good elsewhere;

(vii) *User-friendliness:* heuristics should be clearly expressed, easy to understand and, most important, easy to use. This implies they should have as few parameters as possible and ideally none;

(viii) *Innovation:* preferably, the metaheuristic's principle and / or the efficiency and effectiveness of the heuristics derived from it should lead to new types of applications.

This list has been completed by the third of us and his collaborators:

(ix) *Generality:* the metaheuristic should lead to good results for a large variety of problems.

(x) *Interactivity:* the metaheuristic should allow the user to incorporate his knowledge in order to improve the resolution process.

(xi) *Multiplicity:* the metaheuristic should be able to present several near optimal solutions among which the user can choose.

As argued there and above, VNS possesses, to a large extend, all of those properties. This has led to heuristics among the very best ones for several problems. Interest in VNS is clearly increasing. This is evidenced by the increasing number of papers on that topic (just a few ten years ago, about a dozen five years ago, and about 50 in 2007). Moreover, the 18th EURO Mini conference held in Tenerife in November 2005 was entirely devoted to VNS. It led to special issues of European journal of Operational research and Journal of Heuristics (both forthcoming) and IMA Journal of Management Mathematics in 2007. In retrospect it appears that good healths of the VNS research program is due to the two following decisions, strongly influence by Karl Popper's philosophy of science (Popper [1959]): (i) in devising heuristics favor insight over efficiency (which come later) and (ii) learn from the heuristics mistakes.

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