Finite Dominating Set for the p-Facility Cent-Dian Network Location Problem

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Abstract

A dominating set for a location problem is a set of points that contains an optimal solution for all instances of the problem. The p-facility location problems on a network appear when the possible selections for locating the facilities are the sets of p points of the network. Hooker, Garfinkel and Chen [4] consider a theoretical result to extend the dominating set for the 1-facility problems to the corresponding p-facility problems, and apply this result to propose a finite dominating set for the p-facility cent-dian problem on a network. The optimal solutions of the cent-dian problems are those minimizing a linear combination of the center and median objective functions. Since it is known that the set of vertices and local centers is a dominating set for the single facility cent-dian problem, they claim that it is also a dominating set for the p-facility cent-dian problem. We show a counterexample for $p = 2$ and give an alternative finite dominating set for the p-facility cent-dian. We also provide a solution method that avoids the exhaustive search in all the sets of p points of this dominating set.

Keywords: Location, Cent-dian, Networks.

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1 Introduction

Facility location deals with the problems of locating one or several facilities in order to optimize some criteria with regard to the users. In this paper we consider the problem of selecting several points of a network in order to minimize a function which is distance dependent with respect to given points of the network. The median and the center problems are two well known problems with numerous possible applications. The first is suitable for locating a facility providing a routine service, by means of minimizing the average distances of users to it. The second is appropriate for emergency services where the objective is to have the furthest users as near as possible to the center.

In many real world problems the objective is a mixture of different, possibly adverse objectives, for example, in locating a fire station one may want to minimize the travelling time to the farthest potential source of a call for service as well as one my try to locate as close as possible to the heavily populated areas. The problem is therefore to minimize both objective functions. Such goal may be mathematically expressed by minimizing a new objective function that is a convex combination of the objective functions of the center and median problems. This multi-objective approach for locating a facility on a network was introduced by Halpern [2], who coined the term **cent-dian** for the points which minimize the convex combinations of the center and median objective functions.

Ever since the seminal paper by Hakimi [1], a thread running through network location theory is the identification of a finite subset of the network that necessarily contains an optimal solution for all the instances of a particular location problem. Since Hakimi [1], it is known that the set of vertices is a finite dominating set for the p-median problem. The set of vertices and local centers (points, in the interior of the edges, that are equidistant and balanced with respect to vertices) is a finite dominating set for the p-center problem; e.g. see Moreno [5]. From Halpern [3], it is known that the set of vertices and local centers of the network is a finite dominating set for the single facility cent-dian problem. Hooker, Garfinkel and Chen [4] consider a theoretical result (lemma 10) which extends the finite dominating sets of the single facility problems to the corresponding

p-facility problems, and apply it to the p-facility cent-dian problem (corollary 9). We show that the set of vertices and local centers of the network it is not dominating for the p-facility cent-dian problem by giving a counterexample with an instance where the only optimal solution does not consits of vertices or local centers. We propose an alternative finite dominating set for the p-facility cent-dian problem and an exact solution procedure that avoids the exhaustive selection of the solution.

Next section provides the basic definitions and notation for the formulation of the p-facility cent-dian problem on network that is derived from the classical p-center and p-median problems. Section 3 includes a very simple counterexample for $p = 2$ in a tree network and the new finite dominating set of points. Last section includes a polynomial exact algorithm for every p.

2 Formulation of the problem.

Let $N = (G, l)$ be a *network* where $G = (E, V)$ is a connected undirected graph and l is a positive length function defined on E. The set V of *vertices* is a finite set of points and E is a finite set of *edges*, each one of them is a continuous and linear set of points joining two vertices. The edge e joining vertices i and j is denoted by $e = [i, j]$. The *length* of an edge $e = [i, j] \in E$ is denoted by $l(e) = l(i, j) = l(j, i)$ and represents the cost of going once through it from one vertex to another to satisfy the demand of one user.

Every *point* x on edge $e = [i, j]$ is determined by a value t, $0 \leq t \leq l(e)$ which represents the length of the proportion of the edge between x and i; the point x is then denoted by $x = p(e, t) = p([i, j], t)$. The portion of the edge $[i, j]$ between i and x is denoted by $[i, x]$. The *end points* of edge e (also called *extremes* of the edge) are the vertices $i = p(e, 0)$ and $j = p(e, l(e))$; the points $p(e, t)$, for $0 < t < l(e)$, are the *interior* points of e. Let P denote the set of points (vertices and interior points) of the network N. The *insertion* of an interior point $x = p(e, t) = p([i, j], t)$ in network N transforms it in an equivalent network $N.x$ with the same set of points where the set of the vertices of the graph is $V \cup \{x\}$, the set of edges is $E - \{[i,j]\} \cup \{[i,x],[x,j]\}$ and the length function for the two new edges are given by $l(i,x) = t$ and $l(x,j) = l(i,j) - t$.

A path between two vertices i and j is a minimal sequence of edges of N joining i and j. The length of a path is equal to the sum of the lengths of all its edges. The distance $d(i,j)$ between any two vertices i and j is equal to the length of the shortest path between them. For any two points x and y on N, the paths between them and the distance $d(x,y)$ are defined, respectively, as the paths and the distance between vertices x and y on $(N.x).y$. So P is a mathematical topological space with the topology induced on N by the metric or distance $d(.,.)$ where continuous functions can be defined.

The distance between two points represents the cost of the shortest way of going from one point to the other to supply one user. The distance from a finite set of facility points $X \subset P$ to an user vertex $u \in U$ is given by:

$$d(X,u) = \min_{x \in X} d(x,u).$$

We derive the formulation of the p-facility cent-dian problem from the well known p-center and p-median problems on networks (see [1]). The median problem consists of determining the locations of the set of facilities that minimizes average travel time to or from the facilities, for the population of their users. For a given value of p, the so called p-median problem is to establish p facilities in p potential locations and to supply each user from the established facilities such that the demands of all users are met and the total costs thereby incurred are minimized. The p-center problem is to open p facilities and to assign each user to exactly one of them such that the maximum distance from any open facility to any of the users assigned to it is a minimum.

Given the network $N = (G,l)$ and the set U of vertices where are the users that have to be served, the p-median problem is to find the set $X^* \subset P$, subject to $|X^*| = p$, that minimizes the objective function:

$$f_m(U;X) = \sum_{u \in U} d(X,u).$$

The p-center problem is to find the set $X^* \subset P$, subject to $|X^*| = p$, that minimizes the objective function:

$$f_c(U; X) = \max_{u \in U} d(X, u).$$

Total -or average- distance minimization tends to favor users who are clustered in population centers to the detriment of users who are spatially dispersed. Discrimination of this kind with regard to accessibility may have a severe impact on remote users in the case of an emergency service (ambulances, fire brigades, police cars,...). As a result, the decision maker may want to consider a criterion focusing more on users who get poorly served.

For a given λ, $0 \leq \lambda \leq 1$, the **λ-cent-dian** problem is to find the location that minimizes the objective function defined by: $f_\lambda = \lambda \cdot f_c + (1 - \lambda) \cdot f_m$, where f_c and f_m are the objective functions of the center and median problems, respectively. The value of λ reflects the weight attributed to the maximum distance with respect to the total distance. When $\lambda = 0$, the λ-cent-dian problem is the median problem and when $\lambda = 1$, it is the center problem. For $0 < \lambda < 1$, it can be viewed as a location problem where both efficiency and equity criteria are taken into account; the λ-cent-dian is also a location that minimizes a linear combination of the average and maximum distances to the user vertices.

Given the network N and the set of user vertices U, the single facility λ-cent-dian problem consists in finding the point $x^* \in P$ such that:

$$f_\lambda(U; \{x^*\}) = \min_{x \in P} f_\lambda(U; \{x\}).$$

The p-facility λ-cent-dian problem or the **p-λ-cent-dian problem** consists in finding the set $X^* \subset P$, subject to $|X^*| = p$, that minimizes the objective function

$$f_\lambda(U; X) = \lambda \cdot f_c(U; X) + (1 - \lambda) \cdot f_m(U; X).$$

i.e., such that

$$f_\lambda(U; X^*) \leq f_\lambda(U; X), \forall X \subset P, \text{ with } |X| = p.$$

3 The new finite dominating set.

The following two sets of interior points of the edges of the networks are used in the finite dominating sets for the p-facility location problems on networks.

- A point $x \in P$ is an **extreme point** with *range* r associated to user vertex $u \in U$ (we denote $x \in EP(r; u)$) if x is an interior point of an edge $[i, j]$ such that:

  $$r = d(x, u) = l(x, i) + d(i, u)$$

  or

  $$r = d(x, u) = l(x, j) + d(j, u).$$

- A point $x \in P$ is a **local center** with *range* r associated to user vertices $u, v \in U$ (we denote $x \in LC(r; u, v)$) if x is an interior point of an edge $[i, j]$ such that:

  $$r = d(x, u) = l(x, i) + d(i, u) < l(x, j) + d(j, u)$$

  and

  $$r = d(x, v) = l(x, j) + d(j, v) < l(x, i) + d(i, v).$$

In the standard case the set of user vertices is $U = V$. The set of vertices is a finite dominating set for the p-median problem [1]. The set of vertices and local centers is a finite dominating set for the p-center problem [5] and also for the single facility λ-cent-dian problem [3]. However, it is not a dominating set for the p-λ-cent-dian problem; Figure 1 shows a counterexample for it.

Consider p-λ-cent-dian problem for $p = 2$ and $\lambda = 0.8$ in the tree network with six user vertices given in figure 1; the lengths are shown below each edge. Two clusters of vertices are easily found; the first one with vertices v_1, v_2, v_3 and v_4 and the second one with vertices v_5 and v_6. For the second cluster, the best location is $x_2 = p([v_5, v_6], 5)$ that is the only local center associated to v_5 and v_6; $LC(5; v_5, v_6) = \{x_2\}$. For the first cluster, the local centers associated to vertices v_1, v_2, v_3 and v_4 are: $p([v_1, v_2], 3) \in LC(3; v_1, v_2)$, $p([v_1, v_2], 4) \in LC(4; v_1, v_3) = LC(4; v_1, v_4)$, $p([v_2, v_3], 1) \in LC(1; v_2, v_3)$ and $p([v_2, v_4], 1) \in LC(1; v_2, v_4)$. The best of these candidates is $x_1 = p([v_1, v_2], 4)$. The vertices in the clusters are worse than x_1 or x_2 as candidates for the optimal solution. Other local centers are in the longest edge and clearly they can not be in an optimal solution. The objective value for $X = \{x_1, x_2\}$ is $f_\lambda(X) = 8.0$.

$$f_\lambda(X) = 0.8 \cdot d(x_2, v_5) + 0.2 \cdot \left[\sum_{i=1}^{4} d(x_1, v_i) + \sum_{i=5}^{6} d(x_2, v_i)\right] =$$

$$= 0.8 \cdot 5 + 0.2 \cdot [4 + 2 + 4 + 4 + 5 + 5] = 0.8 \cdot 5 + 0.2 \cdot 24 = 4.0 + 4.8 = 8.8$$

However the optimal solution is $X^* = \{x_1^*, x_2^*\}$, where $x_1^* = p([v_1, v_2], 5)$ and $x_2^* = x_2$, with objective value $f_\lambda(X^*) = 8.4$.

$$f_\lambda(X^*) = 0.8 \cdot d(x_1^*, v_1) + 0.2 \cdot \left[\sum_{i=1}^{4} d(x_1^*, v_i) + \sum_{i=5}^{6} d(x_2^*, v_i)\right] =$$

$$= 0.8 \cdot 5 + 0.2 \cdot [5 + 1 + 3 + 3 + 5 + 5] = 0.8 \cdot 5 + 0.2 \cdot 22 = 4.0 + 4.4 = 8.4$$

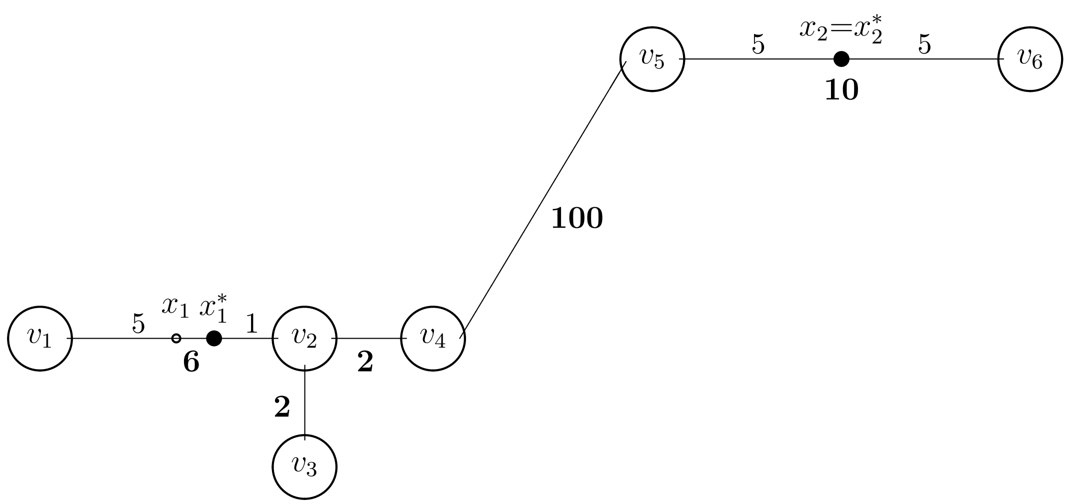

Figure 1: *The counterexample for $p = 2$. The black points constitute the only optimal solution for $\lambda = 0.8$. The point x_1^* is not a local center.*

The proposed finite dominating set for p-facility λ-cent-dian problem consists of the union of three finite sets: the set of vertices, the set of local centers and a finite set of extreme points. Let the *canonical* extreme points be those that have range equal to the distance between two vertices or equal to the range of a local center. Given the network N and the set of user vertices U, let the set of local centers and the set extreme points with range r associated to user vertices, for every r, be:

$$LC(r) = \bigcup_{u,v \in U} LC(r; u, v) \text{ and } EP(r) = \bigcup_{v \in U} EP(r; v).$$

Let the set of distances between vertices and ranges of local centers be given by

$$R = \{r : LC(r) \neq \emptyset\} \cup \{r : r = d(v, u), v \in V, u \in U\}.$$

Then the set of local centers and the set of canonical extreme points associated to user vertices are

$$LC = \bigcup_{r \in R} LC(r) \text{ and } EP = \bigcup_{r \in R} EP(r).$$

The proposed finite dominating set of points is: $D = V \cup LC \cup EP$.

Theorem 1. *The set $D = V \cup LC \cup EP$ of vertices, local centers and canonical extreme points of the network associated to user vertices is a finite dominating set for the p-λ-cent-dian problem.*

Proof. Let every candidate solution be given by a vector $X \in P^p$; i.e., a selection of p facility points $X = (x_1, x_2, ..., x_p)$ with $x_k \in P$, for $k = 1, ..., p$. Every user vertex $u \in U$ is assigned to its closest component of $X \in P^p$, then the result is a series of sets $U(X) = (U_1(X), U_2(X), ..., U_p(X))$ defined by:

$$U_k(X) = \{u \in U : d(x_k, u) = \min_{x \in X} d(x, u)\}.$$

They constitute a partition of the set of user vertices U only if there is not tie; otherwise an optimal partition can be established by solving the ties arbitrarily and then the partition is not unique. The assignment given by the sets $U^*(X) = (U_1^*(X), U_2^*(X), ..., U_p^*(X))$ is an optimal partition for $X \in P^n$ if it is a partition of the set of user vertices that verifies:

$$u \in U_k^*(X) \Rightarrow d(x_k, u) = \min_{x \in X} d(x, u).$$

Given an optimal partition of the user vertex set U, the p-center and the p-median objective functions can be computed by:

$$f_c(U; X) = \max_{k=1,...,p} f_c(U_k^*(X); x_k).$$

$$f_m(U;X) = \sum_{k=1}^{p} f_m(U_k^*(X);x_k).$$

Given a candidate solution X and the corresponding optimal partition $U^*(X)$, associated with each x_k there is a radius $r_k(X)$ which represent the farthest distance from x_k to $U_k^*(X)$:

$$r_k(X) = \max_{u \in U_k^*(X)} d(x_k, u).$$

Then

$$f_c(U;X) = \max_{k=1,\ldots,p} r_k(X) = r^*(X),$$

and

$$f_\lambda(U;X) = \lambda \cdot \max_{k=1,\ldots,p} r_k(X) + (1-\lambda) \cdot f_m(U;X) = \lambda \cdot r^*(X) + (1-\lambda) \cdot f_m(U;X).$$

Assume that $x_k \notin D$ for some $k \in \{1, ..., p\}$. We are going to show that set X can be modified without increasing the function $f_\lambda(U;X)$ until $x_k \in D$ for every k. We consider two cases: $r_k(X) < r^*(X)$ and $r_k(X) = r^*(X)$.

Case 1. Let $r_k(X) < r^*(X)$.

Since $x_k \notin D$, it must be an interior point of an edge $[i, j]$. We are going to show how to move this point x_k on its edge without increasing the function $f_\lambda(U;X)$ until a vertex is reached or $r_k(X)$ equals $r^*(X)$. In order to do so, we will analyse the slope of $f_\lambda(U;X)$ in terms of the distance from x_k to the end vertex i of its edge $[i, j]$.

For any interior point x of an edge $[i, j]$, the set of user vertices which are optimally reached from x through i is denoted by $U^i(x)$; i.e.,

$$U^i(x) = \{u \in U : d(x,u) = l(x,i) + d(i,u) \leq l(x,j) + d(j,u)\}.$$

Analogously,

$$U^j(x) = \{u \in U : d(x,u) = l(x,j) + d(j,u) \leq l(x,i) + d(i,u)\}.$$

Let $U_k^=(X)$ be the set of user vertices that can be assigned to x_k and also to other x_m, for some $m \neq k$; i.e.,

$$U_k^=(X) = \{u \in U : d(x_k, u) = \min_{x \in X} d(x,u) = d(x_m, u), \text{ for some } m \neq k\}.$$

The user vertices assigned to x_k that are no assignable to other facility point are those of $U_k^<(X) = U_k(X) - U_k^=(X)$; i.e.,

$$U_k^<(X) = \{u \in U : d(x_k, u) = \min_{x \in X} d(x, u) < d(x_m, u), \text{ for every } m \neq k\}.$$

The vertices of $U_k^=(X)$ are assignable to x_k and to other facility point in X, but if x_k is moved a small amount ξ towards j or towards i some of these vertices can not remain assigned to x_k and some of them must be assigned only to x_k because the tie is destroyed. Let the new point $x_k(\xi)$ denote x_k when it is moved an amount ξ towards j; i.e., if $x_k = p([i,j],t)$ then $x_k(\xi) = p([i,j], t+\xi)$. Then, those user vertices of $U_k^=(X) \cap U^j(x_k)$ are assigned only to $x_k(\xi)$ and those of $U_k^=(X) - U^j(x_k)$ are no assigned to $x_k(\xi)$.

Let $X(\xi)$ denote the new solution $(x_1, x_2, ..., x_k(\xi), ..., x_p)$. Then the new optimal partition for $X(\xi)$ is given by: $U_m^*(X(\xi)) = U_m^*(X) - [U_k^=(X) \cap U^j(x_k)]$, for every $m \neq k$, and $U_k^*(X(\xi)) = U_k^*(X) - [U_k^=(X) - U^j(x_k)]$.

The slope of $f_m(U; X(\xi))$, as a function of ξ, depends of the user vertices assignable to x_k (those of of $U_k(X(\xi)) = U_k^<(X) \cup U_k^=(X)$) that are optimally reached or not through j. The value of this slope is:

$$s_m^j(\xi) = |U_k^<(X) - U^j(x_k)| - |[U_k^<(X) \cup U_k^=(X)] \cap U^j(x_k)|.$$

When the movement is towards the vertex i the new facility point is $x_k(-\xi) = p(e, t-\xi)$ and, analogously, denoting by $X(-\xi)$ the solution $(x_1, x_2, ..., x_k(-\xi), ..., x_p)$. The slope of $f_m(U; X(-\xi))$, as a function of ξ, is:

$$s_m^i(\xi) = |U_k^<(X) - U^i(x_k)| - |[U_k^<(X) \cup U_k^=(X)] \cap U^i(x_k)|.$$

One of these slopes is not positive since:

$$s_m^i(\xi) + s_m^j(\xi) = -2|[U_k^<(X) \cup U_k^=(X)] \cap U^i(x_k) \cap U^j(x_k)| \leq 0.$$

Let us assume (the other case is similar) that $s_m^j(\xi) \leq 0$.

The slope $s^j_m(\xi)$ could change from non-positive to positive only if one of the following cases hold: a) the set $U^<_k(X) - U^j(x_k)$ gets a vertex, or b) the set $[U^<_k(X) \cup U^=_k(X)] \cap U^j(x_k)$ loses a vertex. Let us analyse them.

a) $U^<_k(X) - U^j(x_k)$ gets a vertex. The only possibilities for this are:

   a1) A vertex leaves $U^j(x_k(\xi))$, but this is not possible because we are moving $x_k(\xi)$ towards j.

   a2) A vertex that is not in $U^j(x_k(\xi))$ comes into $U^<_k(X)$, but this is not possible either because, for this vertex, the distance to $x_k(\xi)$ increases while the distance to x_m, for $m \neq k$, does not change.

b) $[U^<_k(X) \cup U^=_k(X)] \cap U^j(x_k)$ loses a vertex. This is impossible because the distance from $x_k(\xi)$ to the vertices of $U^j(x_k)$ decreases as $x_k(\xi)$ gets closer to j, then they can not leave $U^<_k(X) \cup U^=_k(X)$.

While $r_k(X(\xi)) \leq r^*(X(\xi)) = r^*(X)$ the slope of f_c is zero. Then, the slope of f_λ is: $s^j_\lambda(\xi) = (1-\lambda) \cdot s^j_m(\xi)$, if the movement is towards j, and $s^i_\lambda(\xi) = (1-\lambda) \cdot s^i_m(\xi)$, if the movement is towards i. The sum or these two values is:

$$s^j_\lambda(\xi) + s^i_\lambda(\xi) = \underbrace{(1-\lambda)}_{\geq 0}\underbrace{(s^i_m(\xi) + s^j_m(\xi))}_{\leq 0} \leq 0.$$

So, in any case one of the slopes is no positive. This means that we can always move x_k in the direction which has no positive slope where f_λ does not increase, until it reaches a vertex or the corresponding radius r_k equals r^*. In this last case we will be in case 2.

Case 2. Let $r_k(X) = r^*(X)$.

We study two subcases: $2a$ and $2b$. In case $2a$) $r_m(X) < r_k(X)$, for all $m \neq k$; i.e., $r^*(X) = \max_{i \in \{1,..,p\}} r_i(X)$ is equal only to $r_k(X)$. And in case $2b$) $r_m(X) = r_k(X)$, for some $m \neq k$; i.e., $r^*(X) = \max_{i \in \{1,..,p\}} r_i(X)$ is equal to several of $r_i(X)$, $i \in \{1,..,p\}$.

<u>Case $2a$)</u> $r_m(X) < r_k(X) = r^*(X)$, for all $m \neq k$.

As in case 1, let us analyse the slope of function f_λ when moving $x_k \in [i,j]$ an amount ξ towards j or towards i. Let $x_k(\xi)$ denote x_k when it is moved an amount ξ towards j and $x_k(-\xi)$ denote x_k when it is moved an amount ξ towards i.

Let $U^* = \{u \in U_k^*(X) : d(x_k, u) = r^*(X)\}$. Since $x_k \notin D$, it is not a local center then only the following two cases are possible:

a) $U^* \subset U^j(x_k)$ and $U^* \cap U^i(x_k) = \emptyset$ then $r^*(X(\xi)) = r_k(X(\xi)) = r_k(X) - \xi$.

b) $U^* \subset U^i(x_k)$ and $U^* \cap U^j(x_k) = \emptyset$ then $r^*(X(\xi)) = r_k(X(\xi)) = r_k(X) + \xi$.

Therefore the slope of $f_c(X(\xi))$ is $+1$ or it is -1, for both positive and negative ξ. Thus the slope of f_λ is exppresed in one of the following ways:

a) $s_\lambda(\xi) = s_\lambda^j(\xi) = -\lambda + \, (1-\lambda)s_m^j(\xi)$ and $s_\lambda(-\xi) = s_\lambda^i(\xi) = +\lambda + \, (1-\lambda)s_m^i(\xi)$.

b) $s_\lambda(\xi) = s_\lambda^j(\xi) = +\lambda + \, (1-\lambda)s_m^j(\xi)$ and $s_\lambda(-\xi) = s_\lambda^i(\xi) = -\lambda + \, (1-\lambda)s_m^i(\xi)$.

In both cases, as can be deduced from the analysis of case 1, the sum of the slope when moving x_k towards j plus the slope when moving it towards i is:

$$s_\lambda^i(\xi) + s_\lambda^j(\xi) = \underbrace{(1-\lambda)}_{\geq 0}\underbrace{(s_m^i(\xi) + s_m^j(\xi))}_{\leq 0} \leq 0.$$

Therefore, moving x_k in one of the directions (towards i or towards j) the function f_λ does not increase until $x_k(\xi)$ becomes a vertex (i or j), or a local center where the slope of $f_c(U_k(X(\xi)); x_k(\xi))$ could be $+1$ in both directions, until or $r_k(X(\xi))$ becomes equal to some r_m, with $m \neq k$. In this last case we will be in case $2b$).

Case $2b$) $r_m(X) = r_k(X) = r^*(X)$, for some $m \neq k$.

Let K be the set of indices where $r^*(X)$ is reached; i.e., $k \in K$ if and only if $r_k(X) = r^*(X)$. If for some $k^* \in K$, $x_{k^*} \in LC(r^*) \cup V$ then $r^* \in R$ and $x_k \in EP(r^*)$ for all $k \in K$. Otherwise $x_k \notin CL(r^*) \cup V$ for every $k \in K$. Let us consider this case.

For every $k \in K$, the facility point x_k is interior to an edge and $x_k \notin CL(r^*)$. By the analysis of case $2a$, we can denote the edge containing x_k by $[i_k, j_k]$ in such way that the slope of $f_c(U_k(X); x_k)$ is $+1$ when moving x_k towards j_k and it is -1 when moving x_k

towards i_k (the interchange of the names i_k and j_k could be necessary). We will move at the same time all the points x_k, $k \in K$, an amount ξ on their edges in the direction in which simultaneously every r_k increases or decreases. The slope of the objective function f_λ is:

a) If every x_k, for $k \in K$, is moved ξ towards j_k:

$$s_\lambda(\xi) = +\lambda \; + \; (1-\lambda)\sum_{k \in K} s_k^{j_k}(\xi).$$

b) If every x_k, for $k \in K$, is moved ξ towards i_k:

$$s_\lambda(-\xi) = -\lambda \; + \; (1-\lambda)\sum_{k \in K} s_k^{i_k}(\xi).$$

One of these two slopes must be non positive, because the sum of them is:

$$(1-\lambda)\sum_{k \in K} \underbrace{[s_k^{i_k}(\xi) + s_k^{j_k}(\xi)]}_{\leq 0} \leq 0.$$

Therefore, moving at the same time all the x_k, $k \in K$, in the corresponding directions (towards the i_k's or towards the j_k's) the function f_λ does not increase until some x_k becomes a vertex (i_k or j_k) or a local center with range $r_k(X(\xi)) = r^*(X(\xi))$ where the slope of $f_c(U_k(X(\xi)); x_k(\xi))$ is $+1$ in both directions, or until $r_k(X(\xi))$ becomes equal to some r_m, with $m \notin K$. In this last case a new index comes into K and the process is iterated.

This completes the proof. □

4 Algorithms.

The finite dominating set D provides a rudimentary procedure for solving the problem: an exhaustive search in the set of all combinations of p points of D. The complexity of this algorithm depends on the size of D. Let n be the size of the vertex set V and m be the number of edges.

Proposition 1. *The finite dominating set D for the p-facility λ-cent-dian problem on a network N has size $O(n^3m^2)$.*

Proof. We have $D = V \cup LC \cup EP$. There is at most a local center in each edge associated to every pair of vertices, then $|LC| = O(n^2m)$ and $|R| = O(n^2m)$. For every value $r > 0$, there are at most two extreme points with range r in each edge associated to every vertex, then $|EP(r)| = O(nm)$. Therefore $|EP| = O(n^3m^2)$ and $|D| = O(n^3m^2)$ □

The exact algorithm for solving the p-λ-cent-dian problem, based on the exhaustive search in D^p, has complexity $O(n^{3p+1}m^{2p})$ because $|D^p| = O(n^{3p}m^{2p})$ and the objective function is computed with complexity $O(n)$.

The following result is a consequence of the proof of the theorem and allows us to reduce the complexity of the search in the set of candidates.

Proposition 2. *There is an optimal solution X^* for the p-λ-cent-dian problem such that, if $r^* = f_c(U; X^*)$ is the maximum radius of the solution, then every $x^* \in X^*$ is a local center, a vertex, or an extreme point with range r^*. That is, $X^* \subset V \cup LC \cup EP(r^*)$.*

We can reduce the search for an optimal solution to the set of combinations of p points of $D(r)$, for each $r \in R$. Since $|D(r)| = |V \cup LC \cup EP(r)| = O(n^2m)$ for every r, these $|R|$ searches of the combinations of p points imply evaluating $O(n^{2p+2}m^{p+1})$ candidate solutions. The objective function is computed in $O(n)$ time, so the complexity of the exact algorithm is $O(n^{2p+3}m^{p+1})$.

The complexity of the search can be reduced even more by observing that one of the points in X^* has to be a local center or a vertex that determines the value of r^*, and the other $p-1$ points of X^* have to be vertices or extreme points with range r^*.

Proposition 3. *Let X^* be an optimal solution for the p-λ-cent-dian problem and $r^* = f_c(U; X^*)$ be the maximum radius of the solution. Then, there is a point $x^* \in X^*$ such that $x^* \in LC(r^*)$ or $x^* \in V$ with $d(x^*, u) = r^*$ for some vertex $u \in U$; and the other $p-1$ facility points in X^* are vertices or extreme points with range r^*.*

To obtain a procedure based on this result, let DR the set of pairs (*point,range*) given by

$$DR = \{(x, r) : x \in LC(r)\} \cup \{(v, r) : v \in V, r = d(v, u), \text{ for } u \in U\}.$$

For each $(x,r) \in DR$ we only need to search for solutions consisting of x and $p-1$ points in $V \cup E(r)$. $|DR| = O(n^2m)$ since $|LC| = O(n^2m)$ and $|U| = O(n)$. Moreover $|EP(r)| = O(nm)$ for every $r \geq 0$. All this results in $O(n^2m)$ searches in $(V \cup E(r))^{p-1}$ to find candidate sets of p points. Evaluating each candidate set takes a time $O(n)$. Thus the exact algorithm has complexity $O(n^2m \cdot (nm)^{p-1} \cdot n) = O(n^{p+2}m^p)$.

Proposition 4. *Let X^* be an optimal solution for the p-λ-cent-dian problem, X_c^* be an optimal solution for the p-center problem, and X_m^* be an optimal solution for the p-median problem. Let $r^* = r^*(U; X^*)$, $r_c^* = r^*(X_c)$ and $r_m^* = r^*(X_m)$. Then $r_c^* \leq r^* \leq r_m^*$.*

Then, the proposed algorithm is as follows:

Algorithm.

- <u>Step 1.</u> Obtain the list L of pairs (x,r) consisting in a local center x and its radius r. Add to L all the pairs (v,r) where v is a vertex and $r = d(v,u)$ for some $u \in U$.
- <u>Step 2.</u> Compute the p-center X_c and the p-median X_m. Let $r_c = r^*(X_c)$ and $r_m = r^*(X_m)$ be the corresponding maximum radii.
- <u>Step 3.</u> For every $(x,r) \in L$ with $r_c \leq r \leq r_m$ do de following. Obtain the set $EP(r)$ of extreme points with range r. For every selection Y of $p-1$ points of $V \cap EP(r)$ compute $f_\lambda(U; \{x\} \cap Y)$.
- <u>Step 4.</u> Keep the best of the sets of p points evaluated in step 3.

If the underlying graph is a tree then $m = n-1$ and $|CL| = O(n^2)$; then $|R| = O(n^2)$ and $D(r) = O(n^2)$ for every $r \geq 0$. Therefore $|D| = O(n^4)$ and the exhaustive search on D^p takes time $O(n^{4p+1})$ to get the optimal solution. So the complexity of the proposed algorithm in a tree network is $O(n^{2p+1})$.

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