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The p-Facility Ordered Median Problem on Networks

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Abstract

In this paper we deal with the ordered median problem: a family of location problems that allows us to deal with a large number of real situations which does not fit into the standard models of location analysis. Moreover, this family includes as particular instances many of the classical location models. Here, we analyze the p-facility version of this problem on networks and our goal is to study the structure of the set of candidate points to be optimal solutions.

Key Words: Location, networks, finite dominating sets.

AMS subject classification: 90B10, 90B80, 90B85.

1 Introduction

Network location models have been widely studied in the literature as can be seen in textbooks: Handler and P.B. Mirchandani (1979), Mirchandani and R.L. Francis (1990), Daskin (1995), Drezner (1995), Drezner and H.W. Hamacher (2002), and in the Handbook in Operations Research and Management Science (Ball et al. (1995)) devoted to network models.

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Since the seminal paper by Hakimi (1964), much of this work has been devoted to identify finite sets of points where an optimal solution of a problem must belong to. These sets, called *finite dominating sets* (FDS), are very useful in a wide range of optimization problems, in order to restrict the number of possible candidates to be optimal solutions. Hooker et al. (1991) is an excellent paper on this subject that provides characterizations of FDS for a large number of problems of Location Analysis.

In the last years, the study of the ordered median problem has received a special attention by the researchers in Location Theory. In the continuous case, characterizations of the solution set and polynomial time algorithms have been developed in several papers (see Puerto and Fernández (1995), Puerto et al. (1997), Rodríguez-Chía (1998), Ogryczak (1999), Puerto and Fernández (2000), Rodríguez-Chía et al. (2000), Francis et al. (2000), Lozano et al. (2002), Saameño et al. (2003)). On networks, finite dominating sets have been obtained for particular instances of this problem (see Nickel and Puerto (1999), Kalcsics et al. (2002), Kalcsics et al. (2003)). Recently, also the discrete version of this model has been studied (see Domíguez-Marín et al (2003), Domíguez-Marín et al. (2003)).

In this paper, our goal is to study the structure of the set of candidates to be optimal solutions for the p-facility ordered median problem on networks. This problem has been already analyzed in the literature but with restrictive hypotheses on the weights. Therefore, our study focusses on the problem without additional hypotheses. We will show that the structure of any FDS depends on the number of facilities to be located.

Before introducing the problem some notation is needed. Let $N = (G, l)$ denote a network with underlying graph $G = (V, E)$, where the node set is $V = \{v_1, \dots, v_n\}$ (demand points) and the edge set is $E = \{e_1, \dots, e_m\}$. We restrict ourselves to undirected graphs. Therefore, we write the edge that joins the nodes v_i and v_j as $[v_i, v_j] = [v_j, v_i]$ and as $(v_i, v_j) = (v_j, v_i)$ the edge without the nodes.

The *length* of an edge $e \in E$ is denoted by $l(e) = l(v_i, v_j) = l(v_j, v_i)$ and it represents the cost of going once through the edge to satisfy the demand of one user. By $d(v_i, v_j)$, we denote the length of the shortest path between v_i and v_j measured by l.

A point x on an edge $e = [v_i, v_j]$ is determined by a value t, $0 \leq t \leq l(e)$, which represents the length of the proportion of the edge between x and

v_i, the point x is then denoted by

$$x = p(e,t) = p([i,j],t) \quad \text{or} \quad x = p([j,i], l(e) - t). \tag{1.1}$$

The distance from this point to a node $v_k \neq v_i, v_j$ is:

$$d(x, v_k) := d(v_k, x) := \min\{d(v_k, v_i) + t\ ,\ d(v_k, v_j) + l(e) - t\}.$$

The set of all the points of a network (G,l) is denoted by $A(G)$. Notice that $A(G)$ is a metric space induced by G and the edge lengths. The distance from a node to a vector with p components, $X_p = (x_1, \ldots, x_p) \subseteq A(G) \times \ldots \times A(G)$, is given by

$$d(v, X_p) = \min_{i=1,\ldots,p} d(v, x_i).$$

We consider a set of non-negative weights $\{w_1, \ldots, w_n\}$ called w-weights, where each weight w_i is associated to the node v_i and represents the demand intensity of this node. Let $\beta(\cdot)$ be a permutation of the set $\{1, \ldots, n\}$ verifying that

$$w_{\beta(1)} d(v_{\beta(1)}, X_p) \leq \ldots \leq w_{\beta(n)} d(v_{\beta(n)}, X_p). \tag{1.2}$$

The ordered p-median problem can be defined as to minimize any of the following two expressions:

$$F_\lambda(X_p) = \sum_{i=1}^{n} \lambda_i w_{\beta(i)} d(v_{\beta(i)}, X_p) = \sum_{i=1}^{n} \lambda_{\sigma(i)} w_i d(v_i, X_p) \tag{1.3}$$

where $\{\lambda_1, \ldots, \lambda_n\}$ is a set of non-negative weights called λ-weights and $\sigma(\cdot)$ is a permutation of $\{1, \ldots, n\}$ such that $\sigma(i) < \sigma(j)$ if $d(v_i, X_p) \leq d(v_j, X_p)$ for all $i, j \in \{1, \ldots, n\}$. We will say that λ_i is allocated (assigned) to $v_{\beta(i)}$ or equivalently $\lambda_{\sigma(i)}$ is allocated to v_i.

The λ-weights are the parameters that define the objective function and depending on the values of these parameters we can obtain different problems. In fact, the ordered p-median problem allows to model the p-facility versions of the median ($\lambda_i = 1,\ \ \forall i$), center ($\lambda_n = 1, \lambda_i = 0,\ \ \forall i \neq n$), α-centdian ($\lambda_n = 1,\ \lambda_i = \alpha,\ \ \forall i \neq n$), k-centrum ($\lambda_i = 1$, for $i = n-k+1, \ldots, n$ and $\lambda_i = 0$ for $i = 1, \ldots, n-k$), k-trimmed p-mean location model (we omit the $\frac{k}{2}$ smallest and $\frac{k}{2}$ largest weighted distances, to simplify assume k is even, $\lambda_1 = \ldots = \lambda_{\frac{k}{2}} = 0$, $\lambda_{\frac{k}{2}+1} = \ldots = \lambda_{n-\frac{k}{2}} = 1$, $\lambda_{n-\frac{k}{2}+1} =$

$\ldots = \lambda_n = 0$), etcetera. Notice that we do not have any assumption on the monotonicity of the λ-weights, therefore we do not restrict to the convex nor the concave case, see Nickel and Puerto (1999).

In what follows we give an overview of the different FDS obtained for particular instances of the ordered median problem. In order to do that, we will use the sets EQ (equilibrium points) and Y that are defined in Section 3.

Nickel and Puerto (1999) proves that for $\lambda_1 \geq \ldots \geq \lambda_n$ the node set V constitutes an FDS for the ordered p-median problem. For arbitrary non-negative λ-weights, it also obtains that EQ is an FDS for the single-facility ordered median problem.

Kalcsics et al. (2003) studies the ordered p-median problem where the λ-weights are defined as: $\lambda_1 = \ldots = \lambda_k \neq \lambda_{k+1} = \ldots = \lambda_n$, for a fixed k, such that, $1 \leq k < n$. It proves that the set Y is an FDS for this problem.

Kalcsics et al. (2002) gives an FDS for the single facility ordered median problem with general node weights (the w-weights can be negative). Moreover, for the case of directed networks with non-negative w-weights, they show that there is always an optimal solution in V.

However, none of these papers deals with the general case of the ordered p-median problem. In fact, these papers deal with restrictive hypotheses over the λ-weights. Our goal in this paper is to obtain a finite set of candidates to be optimal solutions of the 2-facility ordered median problem when no hypotheses are made on the set of parameters. Besides, we will prove that the structure of this set depends on the number of service facilities to be located for $p > 2$.

The paper is organized as follows, the next section provides the basic properties of the objective function above mentioned. Section 3 includes a finite set of candidates to be optimal solutions for the 2-facility ordered median problem. In section 4, we show that the structure of the finite set of candidates characterized in the previous section cannot be extended to the general problem of locating p facilities. The paper ends with some concluding remarks.

2 Properties

Before characterizing the set of candidates to be optimal solutions for Problem (1.3), we study some properties which give us insights into the structure of the p-facility ordered median objective function. To this end, we define several sets which will be used later.

Let $X_p = (x_1, \ldots, x_p) \in A(G) \times \ldots \times A(G)$ and $x_k \in [v_{i_k}, v_{j_k}]$, for any $k = 1, \ldots, n$, we define the following sets:

$$U_k(X_p) = \{v \in V \ : \ d(v, X_p) = d(v, x_k)\},$$
$$U_k^{=}(X_p) = \{v \in U_k(X_p) \ : \ d(v, x_k) = d(v, x_m) = \min_{i=1,\ldots,p} d(v, x_i), \text{ for some } m \neq k\}.$$
$$U_k^{<}(X_p) = U_k(X_p) \setminus U_k^{=}(X_p)$$
$$\overline{U}_{i_k} = \{v \in U_k(X_p) \ : \ d(v, x_k) = l(x_k, v_{i_k}) + d(v_{i_k}, v) \leq l(x_k, v_{j_k}) + d(v_{j_k}, v)\}$$
$$U_{i_k} = \overline{U}_{i_k} \setminus (U_k^{=}(X_p) \cup \overline{U}_{j_k})$$

Remark 2.1.

- $U_k(X_p)$ is the set of nodes whose demand can be covered optimally by x_k, that is, the set of nodes that can be allocated to x_k.
- $U_k^{=}(X_p)$ is the set of nodes that can be allocated either to x_k or to x_m for some $m \neq k$.
- $U_k^{<}(X_p)$ is the set of nodes allocated to x_k that cannot be allocated to a different service facility.
- $\overline{U}_{i_k}$ is the set of nodes which can be served optimally by x_k through v_{i_k}.
- U_{i_k} is the set of nodes included in $U_k^{<}(X_p)$, such that, their corresponding distances to their service, x_k, increase when x_k is displaced towards v_{j_k}.
- Notice that based on their definitions, it always holds that $U_{i_k} \subseteq \overline{U}_{i_k}$ and $U_{i_k} \cap \overline{U}_{j_k} = \emptyset$. Moreover, U_{i_k}, $\overline{U}_{j_k}$ and $(U_k^{=}(X_p) \cap \overline{U}_{i_k}) \setminus \overline{U}_{j_k}$ constitutes a partition of the set $U_k(X_p)$, that is, these sets are pairwise disjoint and their union is $U_k(X_p)$.

We say that there exist ties in the vector of weighted distances between the services $x_1, \ldots, x_p$ and their demand nodes if there exist $v_k, v_l \in V$ such that $w_k d(v_k, X_p) = w_l d(v_l, X_p)$.

The nodes of $U_k^=(X_p)$ can be allocated to x_k and to some other service of the vector X_p. However, if x_k is moved a small enough amount ξ either towards v_{j_k} or towards v_{i_k}, then: 1) some of these nodes cannot still be assigned to x_k and 2) some of them will be assigned only to x_k because the existing tie is destroyed. Let $x_k(\xi)$ denote the position of x_k when it is moved an amount ξ towards v_{j_k}; i.e., if $x_k = p([v_{i_k}, v_{j_k}], t)$ then $x_k(\xi) = p([v_{i_k}, v_{j_k}], t + \xi)$.

Lemma 2.1. *If a point $x_k \in (v_{i_k}, v_{j_k})$ is moved an amount ξ towards v_{j_k} the contribution to the slope of $F_\lambda(X_p)$ is $\left(\sum_{v_l \in U_{i_k}} w_l \lambda_{\sigma(l)} - \sum_{v_l \in \overline{U}_{j_k}} w_l \lambda_{\sigma(l)}\right)$ provided that no ties exist in the vector of weighted distances and being $\sigma(\cdot)$ defined in (1.3).*

Proof. Since we have assumed that there are no ties in the vector of distances, the weights $\lambda_1, \ldots, \lambda_n$ are not reallocated after moving x_k a small enough amount. This is because the order of the sequence of weighted distances (1.2) does not change.

The nodes of $U_k^=(X_p)$ can be allocated to x_k or to x_m, for some $m \in \{1, 2, ..., p\} \setminus \{k\}$, but if x_k is moved an amount ξ towards j_k, a new point $x_k(\xi)$ will be generated.

1. The nodes of U_{i_k} are still assigned to $x_k(\xi)$ but the distances to $x_k(\xi)$ increase and their contribution to $F_\lambda(X_p)$ has slope $\sum_{v_l \in U_{i_k}} w_l \lambda_{\sigma(l)}$.

2. The nodes of $\overline{U}_{j_k}$ are allocated to $x_k(\xi)$, because the distance from $x_k(\xi)$ to v_{j_k} decreases. Thus, $-\sum_{v_l \in \overline{U}_{j_k}} w_l \lambda_{\sigma(l)}$ is the contribution of $\overline{U}_{j_k}$ to the slope of $F_\lambda(X_p)$.

3. The nodes of $(U_k^=(X_p) \cap \overline{U}_{i_k}) \setminus \overline{U}_{j_k}$ are allocated to x_m for some $m \in \{1, \ldots, p\} \setminus \{k\}$ and their contribution to $F_\lambda(X_p)$ is null.

Hence, the result follows. □

Let $m^{v_{i_k}}$, $m^{v_{j_k}}$ be the slopes of F_λ when x_k is displaced a small enough amount towards the node v_{i_k} and the node v_{j_k} respectively. Lemma 2.2

shows that there is always a direction (either towards v_{ik} or towards v_{jk}) where the value of the objective function of the problem does not get worse when x_k is moved an amount ξ provided that no ties are allowed in the vector of ordered distances.

Lemma 2.2. *If $x_k \in (v_{i_k}, v_{j_k})$, then either $m^{v_{i_k}}$ or $m^{v_{j_k}}$ is non positive provided that no ties exist in the vector of distances.*

Proof. Using the sets of Remark 2.1 we can write down the slopes $m^{v_{i_k}}$ and $m^{v_{j_k}}$ as:

$$m^{v_{i_k}} = \sum_{v_l \in U_{j_k}} w_l \lambda_{\sigma(l)} - \sum_{v_l \in \overline{U}_{i_k}} w_l \lambda_{\sigma(l)}$$

$$m^{v_{j_k}} = \sum_{v_l \in U_{i_k}} w_l \lambda_{\sigma(l)} - \sum_{v_l \in \overline{U}_{j_k}} w_l \lambda_{\sigma(l)}$$

Then:

$$m^{v_{i_k}} + m^{v_{j_k}} = \left(\sum_{v_l \in U_{i_k}} w_l \lambda_{\sigma(l)} + \sum_{v_l \in U_{j_k}} w_l \lambda_{\sigma(l)} \right) - \left(\sum_{v_l \in \overline{U}_{i_k}} w_l \lambda_{\sigma(l)} + \sum_{v_l \in \overline{U}_{j_k}} w_l \lambda_{\sigma(l)} \right).$$

Since $U_{i_k} \subseteq \overline{U}_{i_k}$ and $U_{j_k} \subseteq \overline{U}_{j_k}$ then $m^{v_{i_k}} + m^{v_{j_k}} \leq 0$ and the result follows. □

Remark 2.2. The above lemma implies that there always exists a movement that strictly decreases the objective function except when $m^{v_{i_k}} = m^{v_{j_k}} = 0$. In this case we can move x_k towards v_{i_k} as well as towards v_{j_k} without increasing the objective function.

3 A finite set of candidates

In this section we identify a finite set of candidates to be optimal solutions of the 2-facility ordered median problem. First of all, we recall the concept of equilibrium points (see Nickel and Puerto (1999)). A point $x \in A(G)$ is in equilibrium with range r with respect to node v_k, $v_l \in V$, if: $r = w_k\, d(v_k, x) = w_l\, d(v_l, x)$. It is important to realize that there may exist subedges in equilibrium with respect to two nodes. We denote by EQ the set of nodes V union with the relative boundary of the points in equilibrium.

(This is, the points in equilibrium that are isolated and the extreme points of the subedges in equilibrium.) In short, EQ is the set of equilibrium points.

Theorem 3.1. *Consider the following sets:*

$$\begin{aligned} R &= \{r \,:\, r = w_i d(v_i, y),\, v_i \in V,\, y \in EQ\}, \\ Y(r) &= \{y \in A(G) \,:\, w_i d(v_i, y) = r,\, v_i \in V\} \quad \textit{with } r \in R, \\ Y &= \bigcup_{r \in R} Y(r), \end{aligned}$$

$T =$ $\{X_2 = (x_1, x_2) \in A(G) \times A(G) \,:\, \exists v_r, v_s \in U_1^{<}(X_2)$ *and* $v_{r'}, v_{s'} \in U_2^{<}(X_2)$ *with* $w_r d(v_r, x_1) = w_{r'} d(v_{r'}, x_2)$ *and* $w_s d(v_s, x_1) = w_{s'} d(v_{s'}, x_2)$. *Moreover, if* $w_r = w_{r'}$ *and* $w_s = w_{s'}$, *then the slopes of the functions* $d(v_r, \cdot)$ *and* $d(v_s, \cdot)$, *in the edge that* x_1 *belongs to, must have the same signs at* x_1 *and the slopes of the functions* $d(v_{r'}, \cdot)$ *and* $d(v_{s'}, \cdot)$, *in the edge that* x_2 *belongs to, must have different signs at* x_2 $\}$, *and*

$$F = (EQ \times Y) \cup T \subset A(G) \times A(G). \tag{3.1}$$

The set F *is a finite set of candidates to be optimal solutions of the* 2*-facility ordered median problem in the network* N.

Remark 3.1. The structure of the set F is different from previous FDS which appeared in the literature. Indeed, the set F is itself a set of candidates for optimal solutions because it is a set of pairs of points. That means that we do not have to choose the elements of this set by pairs to enumerate the whole set of candidates. The candidate solutions may be either a pair of points belonging to $EQ \times Y$ or a pair belonging to T, but they never can be one point of Y and another point of any pair in T.

Proof. We will prove that for any pair $X_2 = (x_1, x_2) \notin F$ there exist movements of its elements that transform the pair X_2 into a new pair $X_2^* = (x_1^*, x_2^*) \in F$ without increasing the objective value of Problem (1.3).

Let $X_2 = (x_1, x_2)$ be a candidate to be optimal solutions for the 2-facility ordered median problem. First, we assume that $x_1 \in EQ$, and $x_2 \notin Y$ and $(x_1, x_2) \notin T$. Then x_2 belongs to the subedge (y, y'), such that,

$y, y' \in Y$ are two different and consecutive points of Y on the edge where x_2 belongs to.

Since $x_1 \in EQ$ and $x_2 \notin Y$ then there is no $v_i \in V$, such that,

$$w_i d(v_i, x_1) = w_i d(v_i, x_2).$$

Moreover, the above equality does not hold for any pair (x_1, x) with $x \in (y, y')$. This means that the set $U_2^=(X_2) = \emptyset$, that is, $U_2^<(X_2)$ does not change when we move x_2 in the subedge (y, y'). In addition, since $x_1 \in EQ$ and $x_2 \notin Y$, the vector of ordered weighted distances only can have ties between the elements of the set $U_1^<(X_2)$. Hence, there are no reassignments of the λ-weights after any movement of x_2 in the subedge (y, y'). Therefore, the problem of finding the best location of the second facility in (y, y') is a 1-facility median problem in $U_2^<(X_2)$. Then, an optimal solution always exists on the extreme points of the interval (y, y') and the new pair $X_2' = (x_1, x_2') \in EQ \times Y$ is not worse than (x_1, x_2).

In what follows we analyze the situation where neither x_1 nor x_2 belong to EQ. We distinguish the following four cases:

Case 1: There exist no ties in the vector of weighted distances between the nodes and their service facilities.

Case 2: There exists one tie in the vector of weighted distances between the nodes and their service facilities.

Case 3: There exist two ties in the vector of weighted distances between the nodes and their service facilities.

Case 4: There exist more than two ties in the vector of weighted distances between the nodes and their service facilities.

It is worth noting that in these cases, since neither x_1 nor x_2 belong to EQ, the ties in the vector of weighted distances (see (1.2)) have to occur between weighted distances from two nodes: one associated to x_1 and the other to x_2. Indeed, if there would exist two equal weighted distances between two nodes associated with the same service, this service would be an equilibrium point.

Case 1: Using Lemma 2.2, we can move x_1 or x_2 without increasing the objective value while a tie does not occur.

Case 2: There exist $v_r \in U_1(X_2)$ and $v_{r'} \in U_2(X_2)$ such that $w_r d(v_r, x_1) = w_{r'} d(v_{r'}, x_2)$. First we have that $\{v_r, v_{r'}\} \cap U_1^{=}(X_2) = \emptyset$. Indeed, if $v_r \in U_1^{=}(X_2)$, we would have that $w_r d(v_r, x_1) = w_r d(v_r, x_2)$. However, since we have assumed that $w_r d(v_r, x_1) = w_{r'} d(v_{r'}, x_2)$, it follows that $w_r d(v_r, x_2) = w_{r'} d(v_{r'}, x_2)$, that is, $x_2 \in EQ$, what is impossible because by hypothesis $x_2 \notin EQ$. A similar argument can be used if $v_{r'} \in U_1^{=}(X_2)$.

Assume that x_1 belongs to the edge $[v_{i_1}, v_{j_1}]$ and that x_2 belongs to the edge $[v_{i_2}, v_{j_2}]$. Moreover, we denote by $\lambda_{\sigma(r)}$ and $\lambda_{\sigma(r')}$ the λ-weights assigned to v_r and $v_{r'}$, respectively. We can assume without loss of generality that $\sigma(r') = \sigma(r) + 1$, $v_r \in U_{j_1}$ and $v_{r'} \in U_{j_2}$. For sake of simplicity, we denote by $V_T = \{v_r, v_{r'}\} \cup U_1^{=}(X_2)$.

In this case, if we move x_1, a small enough amount ξ_1, towards v_{j_1} and x_2, a small enough amount ξ_2, towards v_{j_2}, such that $\xi_1 w_r = \xi_2 w_{r'}$, we have that the change in the objective function due to these movements is:

$$m^{v_{j_1}}(\xi_1)+m^{v_{j_2}}(\xi_2)=\xi_1\Big(\sum_{v_t\in U_{i_1}\setminus V_T} w_t\lambda_{\sigma(t)}-\sum_{v_t\in \overline{U}_{j_1}\setminus V_T} w_t\lambda_{\sigma(t)}-\sum_{v_t\in U_1^{=}(X_2)\cap(\overline{U}_{j_1}\setminus\overline{U}_{j_2})} w_t\lambda_{\sigma(t)}-w_r\lambda_{\sigma(r)}\Big)$$

$$+\,\xi_2\Big(\sum_{v_t\in U_{i_2}\setminus V_T} w_t\lambda_{\sigma(t)} -\sum_{v_t\in \overline{U}_{j_2}\setminus V_T} w_t\lambda_{\sigma(t)} - \sum_{v_t\in U_1^{=}(X_2)\cap(\overline{U}_{j_2}\setminus\overline{U}_{j_1})} w_t\lambda_{\sigma(t)} - w_{r'}\lambda_{\sigma(r)+1}\Big)$$

$$+\sum_{v_t\in U_1^{=}(X_2)\setminus(\overline{U}_{j_1}\cup\overline{U}_{j_2})} \min\{\xi_1,\xi_2\}w_t\lambda_{\sigma(t)} - \sum_{v_t\in U_1^{=}(X_2)\cap(\overline{U}_{j_1}\cap\overline{U}_{j_2})} \max\{\xi_1,\xi_2\}w_t\lambda_{\sigma(t)}$$

and if we move the same amounts as before, x_1 and x_2 towards v_{i_1} and v_{i_2} respectively, we have that the change in the objective function due to these movements is:

$$m^{v_{i_1}}(\xi_1) + m^{v_{i_2}}(\xi_2) =$$

$$\xi_1\Big(\sum_{v_t\in U_{j_1}\setminus V_T} w_t\lambda_{\sigma(t)} -\sum_{v_t\in \overline{U}_{i_1}\setminus V_T} w_t\lambda_{\sigma(t)} - \sum_{v_t\in U_1^{=}(X_2)\cap(\overline{U}_{i_1}\setminus\overline{U}_{i_2})} w_t\lambda_{\sigma(t)} + w_r\lambda_{\sigma(r)}\Big)$$

$$+\,\xi_2\Big(\sum_{v_t\in U_{j_2}\setminus V_T} w_t\lambda_{\sigma(t)} -\sum_{v_t\in \overline{U}_{i_2}\setminus V_T} w_t\lambda_{\sigma(t)} - \sum_{v_t\in U_1^{=}(X_2)\cap(\overline{U}_{i_2}\setminus\overline{U}_{i_1})} w_t\lambda_{\sigma(t)} w_t\lambda_{\sigma(t)} + w_{r'}\lambda_{\sigma(r)+1}\Big)$$

$$+\sum_{v_t\in U_1^{=}(X_2)\setminus(\overline{U}_{i_1}\cup\overline{U}_{i_2})} \min\{\xi_1,\xi_2\}w_t\lambda_{\sigma(t)} - \sum_{v_t\in U_1^{=}(X_2)\cap(\overline{U}_{i_1}\cap\overline{U}_{i_2})} \max\{\xi_1,\xi_2\}w_t\lambda_{\sigma(t)}.$$

Since $U_{j_q} \subseteq \overline{U}_{j_q}$, $U_{i_q} \subseteq \overline{U}_{i_q}$ for $q = 1, 2$, $U_1^{=}(X_2) \setminus (\overline{U}_{j_1} \cup \overline{U}_{j_2}) \subseteq U_1^{=}(X_2) \cap (\overline{U}_{i_1} \cap \overline{U}_{i_2})$, $U_1^{=}(X_2) \setminus (\overline{U}_{i_1} \cup \overline{U}_{i_2}) \subseteq U_1^{=}(X_2) \cap (\overline{U}_{j_1} \cap \overline{U}_{j_2})$ then $m^{v_{j_1}}(\xi_1) + m^{v_{j_2}}(\xi_2) + m^{v_{i_1}}(\xi_1) + m^{v_{i_2}}(\xi_2)$ is non positive. Therefore, there exists a movement of x_1 and x_2 that does not increase the objective value.

Notice that the initial assumptions $v_r \in U_{j_1}$ and $v_{r'} \in U_{j_2}$ are not restrictive because if $v_r \in \overline{U}_{j_1} \setminus U_{j_1}$ ($v_{r'} \in \overline{U}_{j_2} \setminus U_{j_2}$) then the term $w_r \lambda_{\sigma(r)}$ ($w_{r'} \lambda_{\sigma(r)+1}$) in the above two expressions would appear with negative sign.

Case 3: There exist $v_r, v_s \in U_1(X_2)$ and $v_{r'}, v_{s'} \in U_2(X_2)$ such that $w_r d(v_r, x_1) = w_{r'} d(v_{r'}, x_2)$ and $w_s d(v_s, x_1) = w_{s'} d(v_{s'}, x_2)$. First, notice that using the arguments of Case 2, we obtain that $\{v_r, v_{r'}, v_s, v_{s'}\} \cap U_1^{=}(X_2) = \emptyset$.

We distinguish two subcases:

Case 3.1: If $w_r \neq w_{r'}$ or $w_s \neq w_{s'}$ the pairs in this subcase are included in T and therefore belong to the the set of candidates to be optimal solutions.
Case 3.2: If $w_r = w_{r'}$ and $w_s = w_{s'}$, that is, $d(v_r, x_1) = d(v_{r'}, x_2)$ and $d(v_s, x_1) = d(v_{s'}, x_2)$. We will distinguish four more subcases. Before that, in order to obtain an easy understanding, we assume without loss of generality that:

i) $x_1 \in [v_{i_1}, v_{j_1}]$, $\sigma(r') = \sigma(r) + 1$.

ii) $x_2 \in [v_{i_2}, v_{j_2}]$, $\sigma(s') = \sigma(s) + 1$.

Let $V_T = \{v_r, v_{r'}, v_s, v_{s'}\} \cup U_1^{=}(X_2)$. After that, we proceed with the four subcases:

3.2.1 The slopes of the functions $d(v_r, \cdot)$ and $d(v_s, \cdot)$ on the edge $[v_{i_1}, v_{j_1}]$ have the same sign at x_1 and the slopes of the functions $d(v_{r'}, \cdot)$ and $d(v_{s'}, \cdot)$ on the edge $[v_{i_2}, v_{j_2}]$ have the same sign at x_2.

3.2.2 The slopes of the functions $d(v_r, \cdot)$ and $d(v_s, \cdot)$ on the edge $[v_{i_1}, v_{j_1}]$ have different sign at x_1 and the slopes of the functions $d(v_{r'}, \cdot)$ and $d(v_{s'}, \cdot)$ on the edge $[v_{i_2}, v_{j_2}]$ have different sign at x_2.

3.2.3 The slopes of the functions $d(v_r, \cdot)$ and $d(v_s, \cdot)$ on the edge $[v_{i_1}, v_{j_1}]$ have different sign at x_1 and the slopes of the functions $d(v_{r'}, \cdot)$ and $d(v_{s'}, \cdot)$ on the edge $[v_{i_2}, v_{j_2}]$ have the same sign at x_2.

3.2.4 The slope of the function $d(v,\cdot)$ for some $v \in \{v_r, v_{r'}, v_s, v_{s'}\}$ is not defined at the service facility that covers v.

It should be noted that any other configuration reduces to the previous ones interchanging the name of the points x_1 and x_2.

Now, we prove that there exists a movement for the first two cases where the objective value is not increased.

3.2.1 Since the sign of the slopes of the functions $d(v_r,\cdot)$ and $d(v_s,\cdot)$ at x_1 are the same, we can assume without loss of generality that $v_r, v_s \in U_{j_1}$. In the same way, we assume that $v_{r'}, v_{s'} \in U_{j_2}$.

If we move, the same small enough amount, x_1 and x_2 towards v_{j_1} and v_{j_2} respectively, we have that the slope of these movements is

$$\begin{aligned} m^{v_{j_1}} + m^{v_{j_2}} = & \sum_{v_t \in U_{i_1} \setminus V_T} w_t \lambda_{\sigma(t)} - \sum_{v_t \in \overline{U}_{j_1} \setminus V_T} w_t \lambda_{\sigma(t)} + \sum_{v_t \in U_{i_2} \setminus V_T} w_t \lambda_{\sigma(t)} - \sum_{v_t \in \overline{U}_{j_2} \setminus V_T} w_t \lambda_{\sigma(t)} \\ & - \sum_{v_t \in U_1^{=}(X_2) \cap (\overline{U}_{j_1} \cup \overline{U}_{j_2})} w_t \lambda_{\sigma(t)} + \sum_{v_t \in U_1^{=}(X_2) \setminus (\overline{U}_{j_1} \cup \overline{U}_{j_2})} w_t \lambda_{\sigma(t)} \\ & - w_r \lambda_{\sigma(r)} - w_{r'} \lambda_{\sigma(r)+1} - w_s \lambda_{\sigma(s)} - w_{s'} \lambda_{\sigma(s)+1}, \end{aligned}$$

and if we move by the same amount, x_1 and x_2 towards v_{i_1} and v_{i_2} respectively, we have that the slope of these movements is

$$\begin{aligned} m^{v_{i_1}} + m^{v_{i_2}} = & \sum_{v_t \in U_{j_1} \setminus V_T} w_t \lambda_{\sigma(t)} - \sum_{v_t \in \overline{U}_{i_1} \setminus V_T} w_t \lambda_{\sigma(t)} + \sum_{v_t \in U_{j_2} \setminus V_T} w_t \lambda_{\sigma(t)} - \sum_{v_t \in \overline{U}_{i_2} \setminus V_T} w_t \lambda_{\sigma(t)} \\ & - \sum_{v_t \in U_1^{=}(X_2) \cap (\overline{U}_{i_1} \cup \overline{U}_{i_2})} w_t \lambda_{\sigma(t)} + \sum_{v_t \in U_1^{=}(X_2) \setminus (\overline{U}_{i_1} \cup \overline{U}_{i_2})} w_t \lambda_{\sigma(t)} \\ & + w_r \lambda_{\sigma(r)} + w_{r'} \lambda_{\sigma(r)+1} + w_s \lambda_{\sigma(s)} + w_{s'} \lambda_{\sigma(s)+1}. \end{aligned}$$

Hence, since $U_{j_q} \subseteq \overline{U}_{j_q}$, $U_{i_q} \subseteq \overline{U}_{i_q}$ for $q = 1, 2$; $U_1^{=}(X_2) \setminus (\overline{U}_{j_1} \cup \overline{U}_{j_2}) \subseteq U_1^{=}(X_2) \cap (\overline{U}_{i_1} \cup \overline{U}_{i_2})$ and $U_1^{=}(X_2) \setminus (\overline{U}_{i_1} \cup \overline{U}_{i_2}) \subseteq U_1^{=}(X_2) \cap (\overline{U}_{j_1} \cup \overline{U}_{j_2})$, we have that $m^{v_{j_1}} + m^{v_{j_2}} + m^{v_{i_1}} + m^{v_{i_2}}$ is non positive. Therefore, at least one of these two movements cannot increase the value of the objective function.

Notice that using the arguments of Case 2, the initial assumptions $v_r, v_s \in U_{j_1}$ and $v_{r'}, v_{s'} \in U_{j_2}$ are not restrictive.

3.2.2 Since the sign of the slopes of the functions $d(v_r,\cdot)$ and $d(v_s,\cdot)$ at x_1 are different we can assume without loss of generality that $v_r \in U_{j_1}$ and $v_s \in U_{i_1}$. In the same way, we assume that $v_{r'} \in U_{j_2}$ and $v_{s'} \in U_{i_2}$.

If we move x_1 and x_2 by the same small enough amount, towards v_{j_1} and v_{j_2} respectively, we have that the slope of these movements is

$$\begin{aligned} m^{v_{j_1}} + m^{v_{j_2}} = & \sum_{v_t \in U_{i_1}\setminus V_T} w_t\lambda_{\sigma(t)} - \sum_{v_t \in \overline{U}_{j_1}\setminus V_T} w_t\lambda_{\sigma(t)} + \sum_{v_t \in U_{i_2}\setminus V_T} w_t\lambda_{\sigma(t)} - \sum_{v_t \in \overline{U}_{j_2}\setminus V_T} w_t\lambda_{\sigma(t)} \\ & - \sum_{v_t \in U_1^{=}(X_2)\cap(\overline{U}_{j_1}\cup\overline{U}_{j_2})} w_t\lambda_{\sigma(t)} + \sum_{v_t \in U_1^{=}(X_2)\setminus(\overline{U}_{j_1}\cup\overline{U}_{j_2})} w_t\lambda_{\sigma(t)} \\ & - w_r\lambda_{\sigma(r)} - w_{r'}\lambda_{\sigma(r)+1} + w_s\lambda_{\sigma(s)} + w_{s'}\lambda_{\sigma(s)+1}. \end{aligned}$$

Besides, if we move the same amount, x_1 and x_2 towards v_{i_1} and v_{i_2} respectively, we have that the slope of these movements is

$$\begin{aligned} m^{v_{i_1}} + m^{v_{i_2}} = & \sum_{v_t \in U_{j_1}\setminus V_T} w_t\lambda_{\sigma(t)} - \sum_{v_t \in \overline{U}_{i_1}\setminus V_T} w_t\lambda_{\sigma(t)} + \sum_{v_t \in U_{j_2}\setminus V_T} w_t\lambda_{\sigma(t)} - \sum_{v_t \in \overline{U}_{i_2}\setminus V_T} w_t\lambda_{\sigma(t)} \\ & - \sum_{v_t \in U_1^{=}(X_2)\cap(\overline{U}_{i_1}\cup\overline{U}_{i_2})} w_t\lambda_{\sigma(t)} + \sum_{v_t \in U_1^{=}(X_2)\setminus(\overline{U}_{i_1}\cup\overline{U}_{i_2})} w_t\lambda_{\sigma(t)} \\ & + w_r\lambda_{\sigma(r)} + w_{r'}\lambda_{\sigma(r)+1} - w_s\lambda_{\sigma(s)} - w_{s'}\lambda_{\sigma(s)+1}. \end{aligned}$$

Hence, since $U_{j_q} \subseteq \overline{U}_{j_q}$, $U_{i_q} \subseteq \overline{U}_{i_q}$ for $q = 1, 2$; $U_1^{=}(X_2)\setminus(\overline{U}_{j_1}\cup\overline{U}_{j_2}) \subseteq U_1^{=}(X_2)\cap(\overline{U}_{i_1}\cup\overline{U}_{i_2})$ and $U_1^{=}(X_2)\setminus(\overline{U}_{i_1}\cup\overline{U}_{i_2}) \subseteq U_1^{=}(X_2)\cap(\overline{U}_{j_1}\cup\overline{U}_{j_2})$, we have that $m^{v_{j_1}} + m^{v_{j_2}} + m^{v_{i_1}} + m^{v_{i_2}}$ is non positive. Therefore, at least one of these two movements cannot increase the objective function.

Notice that using the arguments of Case 2, the initial assumptions $v_r \in U_{j_1}$, $v_s \in U_{i_1}$, $v_{r'} \in U_{j_2}$ and $v_{s'} \in U_{i_2}$ are not restrictive.

3.2.3 The pairs in this subcase are included in T and therefore belong to the the set of candidates to be optimal solutions (see Example 3.2).

3.2.4 Without loss of generality assume that the slope of the function $d(v_r,\cdot)$ is not defined at x_1 when $v_r \in \overline{U}_{j_1} \cap \overline{U}_{i_1}$ (the distance $d(v_r,\cdot)$ has a breakpoint at x_1). In this case, if we move x_1 and x_2 as in 3.2.1 or 3.2.2, we have that the expressions of $m^{v_{j_1}} + m^{v_{j_2}}$ and $m^{v_{i_1}} + m^{v_{i_2}}$ are equal to the ones obtained in cases 3.2.1 or 3.2.2, respectively (depending of the relative position of the nodes $\{v_{r'}, v_s, v_{s'}\}$

and their corresponding service facility), except the term $w_r\lambda_{\sigma(r)}$ that appears in these two expressions with negative sign. Therefore, $m^{v_{j_1}} + m^{v_{j_2}} + m^{v_{i_1}} + m^{v_{i_2}}$ is again non positive. A similar argument can be used when more than one of the slopes of the distance functions are not defined.

Case 4: There exist $v_{r_1},\ldots,v_{r_Q} \in U_1(X_2)$ and $v_{r'_1},\ldots,v_{r'_Q} \in U_2(X_2)$, with $Q > 2$, such that $w_{r_l}d(v_{r_l},x_1) = w_{r'_l}d(v_{r'_l},x_2)$ and $w_{r_l} = w_{r'_l}$ for $l = 1,\ldots,Q$. (Notice that, if $w_{r_l} \neq w_{r'_l}$ for some $l = 1,\ldots,Q$, we are in a particular instance of Case 3.1 and the pair (x_1,x_2) belongs to T.) We assume that x_1 belongs to the edge $[v_{i_1},v_{j_1}]$ and that x_2 belongs to the edge $[v_{i_2},v_{j_2}]$. We distinguish two subcases:

4.1 There exist no $v_{r_{l_c}}, v_{r_{l_d}} \in U_1(X_2)$ with $l_c, l_d \in \{1,\ldots,Q\}$ and $v_{r'_{l_c}}, v_{r'_{l_d}} \in U_2(X_2)$ such that the slopes of the functions $d(v_{r_{l_c}},\cdot)$ and $d(v_{r_{l_d}},\cdot)$ on the edge $[v_{i_1},v_{j_1}]$ have different signs at x_1 and the slopes of the functions $d(v_{r'_{l_c}},\cdot)$ and $d(v_{r'_{l_d}},\cdot)$ on the edge $[v_{i_2},v_{j_2}]$ have the same sign at x_2.

4.2 There exist $v_{r_{l_c}}, v_{r_{l_d}} \in U_1(X_2)$ with $l_c, l_d \in \{1,\ldots,Q\}$ and $v_{r'_{l_c}}, v_{r'_{l_d}} \in U_2(X_2)$ such that the slopes of the functions $d(v_{r_{l_c}},\cdot)$ and $d(v_{r_{l_d}},\cdot)$ on the edge $[v_{i_1},v_{j_1}]$ have different signs at x_1 and the slopes of the functions $d(v_{r'_{l_c}},\cdot)$ and $d(v_{r'_{l_d}},\cdot)$ on the edge $[v_{i_2},v_{j_2}]$ have the same sign at x_2.

In the first case, the four nodes defining each two ties in the sequence of ordered weighted distances are either in case 3.2.1 or 3.2.2. Therefore, using the same arguments as in 3.2.1 and 3.2.2, there exits a movement of x_1 and x_2 that does not get a worse objective value. The second case is a particular instance of the Case 3.2.3 and X_2 is again included in the set T.

We have proved that in all the cases $m^{v_{i_1}} + m^{v_{i_2}} + m^{v_{j_1}} + m^{v_{j_2}} \leq 0$ when $X_2 = (x_1,x_2) \notin F$. Thus, if $m^{v_{i_1}} + m^{v_{i_2}}$ or $m^{v_{j_1}} + m^{v_{j_2}}$ are different from zero there exists a movement of $X_2 = (x_1,x_2)$ to a new pair X'_2 which strictly decreases the objective value. Otherwise, if $m^{v_{i_1}} + m^{v_{i_2}} = m^{v_{j_1}} + m^{v_{j_2}} = 0$ then the movements of x_1 and x_2, respectively, towards v_{i_1} and v_{i_2} as well as towards v_{j_1} and v_{j_2} do not increase the objective value.

One of these two displacements avoid cycling since one of them has not been used in the opposite direction in the previous step (see Remark 2.2).

The movement from X_2 to X'_2 is valid whenever the sets $U_k(X'_2)$, $U_k^{=}(X'_2)$, $U_k^{<}(X'_2)$, $\overline{U}_{i_k}$ and $\overline{U}_{j_k}$ (associated to X'_2) for $k = 1, 2$ do not change. Hence, if the maximal displacement without increasing the objective value transforms X_2 into X''_2 and $X''_2 \notin F$, we repeat the same process a finite number of times until a pair $X^*_2 \in F$ is reached. □

The following examples show that the set F can not be shrunk because even in easy cases on the real line all the points are needed. The first example shows a graph where the optimal solution $X_2 = (x_1, x_2)$ verifies that x_1 is an equilibrium point and x_2 is not a equilibrium point which belongs to $Y(r) \setminus EQ$ for a given r. In the second example the optimal solution $X_2 = (x_1, x_2)$ belongs to the set T.

Example 3.1. Let $N = (G, l)$ be a network with underlying graph $G = (V, E)$ where $V = \{v_1, v_2, v_3, v_4\}$ and $E = \{[v_1, v_2], [v_2, v_3], [v_3, v_4]\}$. The length function is given by $l([v_1, v_2]) = 3, l([v_2, v_3]) = 20, l([v_3, v_4]) = 6$. The w-weights are all equal to one and the λ-weights are $\lambda_1 = 0.1, \lambda_2 = 0.2, \lambda_3 = 0.4, \lambda_4 = 0.3$, see Figure 1.

It should be noted that this example has not an optimal solution on the edge $[v_2, v_3]$ because any point of this edge is dominated by v_2 or v_3. In addition, using the symmetry of the problem we have omitted the evaluation of some of the elements of Y.

In Figure 1 we represent the nodes (dots), the equilibrium points (ticks) and elements of Y (small ticks). Notice that in this case there are no pairs in T.

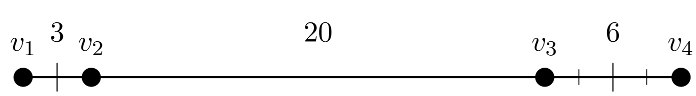

Figure 1: *Illustration of Example 3.1*

In this example the optimal solution is given by $x_1 = p([v_1, v_2], 1.5)$ and $x_2 = p([v_3, v_4], 1.5)$ (see Table 1). It is easy to check that x_1 is an equilibrium point between v_1 and v_2, and $x_2 \in Y(1.5)$. It is worth noting that the radius 1.5 is given by the distance from the equilibrium point,

Candidate pair X_2	Value	Candidate pair X_2	Value
$p([v_1,v_2],0),p([v_3,v_4],0)$	3	$p([v_1,v_2],1.5),p([v_3,v_4],0)$	2.7
$p([v_1,v_2],0),p([v_3,v_4],1.5)$	2.85	$p([v_1,v_2],1.5),p([v_3,v_4],1.5)$	2.4
$p([v_1,v_2],0),p([v_3,v_4],3)$	2.7	$p([v_1,v_2],1.5),p([v_3,v_4],3)$	2.55

Table 1: *Evaluation of the candidate pairs of Example 3.1*

$p([v_1,v_2],1.5)$, generated by v_1 and v_2 to any of these nodes.

Example 3.2. Let $N=(G,l)$ be a network with underlying graph $G=(V,E)$ where $V=\{v_1,v_2,v_3,v_4,v_5\}$ and $E=\{[v_1,v_2],[v_2,v_3],[v_3,v_4],[v_4,v_5]\}$. The length function is given by $l([v_1,v_2])=5, l([v_2,v_3])=20, l([v_3,v_4])=5.1, l([v_4,v_5])=1$. The w-weights are all equal to one and the λ-weights are $\lambda_1=0,\lambda_2=1,\lambda_3=0,\lambda_4=1,\lambda_5=1.1$, see Figure 2.

In Figure 2, we use the same notation as in Figure 1 and pairs of T are represented by ($\star$). By domination and symmetry arguments not all the candidates are necessary and therefore, they are not depicted.

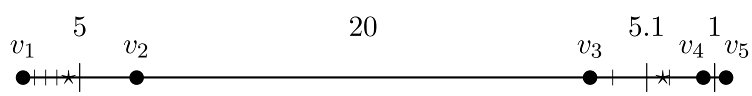

Figure 2: *Illustration of Example 3.2*

In this example the optimal solution is given by $x_1=p([v_1,v_2],2)$ and $x_2=p([v_3,v_4],3.1)$ (see Table 2). Therefore the optimal pair (x_1,x_2) belongs to the set T. Indeed, $d(v_1,x_1)=d(v_4,x_2)$ and $d(v_2,x_1)=d(v_5,x_2)$ and the slopes of $d(v_1,\cdot),d(v_2,\cdot)$ in the edge $[v_1,v_2]$ at x_1 are $1,-1$ respectively; and the slopes of $d(v_4,\cdot),d(v_5,\cdot)$ in the edge $[v_3,v_4]$ at x_2 are $-1,-1$ respectively.

Once we have proved that F is an essential set to describe the set of optimal solutions of the 2-facility ordered median problem we want to know its cardinality.

Proposition 3.1. *The cardinality of F is $O(m^3n^6)$.*

Proof. In each edge there are at most two equilibrium points associated to each pair of nodes. Thus $|EQ|=O(mn^2)$ and $|R|=O(mn^3)$. The

Candidate pair X_2	Value	Candidate pair X_2	Value
$p([v_1,v_2],0),p([v_3,v_4],0)$	11.81	$p([v_1,v_2],2.05),p([v_3,v_4],3.05)$	8.455
$p([v_1,v_2],0),p([v_3,v_4],2.55)$	11.6	$p([v_1,v_2],2.45),p([v_3,v_4],2.55)$	9.005
$p([v_1,v_2],0),p([v_3,v_4],3.05)$	10.6	$p([v_1,v_2],2.5),p([v_3,v_4],0)$	14.31
$p([v_1,v_2],0),p([v_4,v_5],0)$	10.61	$p([v_1,v_2],2.5),p([v_3,v_4],2.5)$	9.06
$p([v_1,v_2],0),p([v_4,v_5],0.5)$	11.66	$p([v_1,v_2],2.5),p([v_3,v_4],2.55)$	8.955
$p([v_1,v_2],0),p([v_4,v_5],1)$	11.71	$p([v_1,v_2],2.5),p([v_3,v_4],2.6)$	8.95
$p([v_1,v_2],0.5),p([v_4,v_5],0.5)$	11.16	$p([v_1,v_2],2.5),p([v_3,v_4],3.05)$	8.905
$p([v_1,v_2],1),p([v_4,v_5],0)$	10.61	$p([v_1,v_2],2.5),p([v_3,v_4],3.6)$	8.96
$p([v_1,v_2],1),p([v_4,v_5],1)$	11.71	$p([v_1,v_2],2.5),p([v_4,v_5],0)$	9.11
$p([v_1,v_2],1.45),p([v_3,v_4],2.55)$	10.005	$p([v_1,v_2],2.5),p([v_4,v_5],0.5)$	9.16
$p([v_1,v_2],1.95),p([v_3,v_4],3.05)$	8.455	$p([v_1,v_2],2.5),p([v_4,v_5],1)$	10.21
$p([v_1,v_2],2),p([v_3,v_4],3.1)$	8.41		

Table 2: *Evaluation of the candidate pairs of Example 3.2*

maximum degree of a node $v_i \in V$ is m (the star network) so $|Y(r)| = O(mn)$ with $r \in R$. Thus, $|Y| = O(m^2n^4)$. On the second hand, on each edge, each pair of nodes may determine an element of a pair in T. Therefore, the set T has a cardinality $O((n^2m)^2)$. In conclusion $|F| = O(m^3n^6 + m^2n^4) = O(m^3n^6)$. □

It is worth noting that F is an actual set of finite elements to be optimal solutions of Problem (1.3). The difference with previous approaches is that this set is not a set of candidates for each individual facility but it is the set of candidate pairs to be optimal solutions.

4 A discouraging result for the p-facility case

It is well-known that FDS of polynomial size exist for the classical p-median, p-center and p-centdian problems (see Hooker et al. (1991)). In the previous section we have found a finite set of candidates to be optimal solutions of the 2-facility ordered median problem in a network. However, despite the similarity existing between those problems and the p-facility ordered median problem, these results can not be extended to our model.

The reason for this is the following. For the 1-facility ordered median problem we have that the set of candidates to be optimal solutions is EQ, that means, the equilibrium points (see Nickel and Puerto (1999)). For the 2-facility ordered median problem we have obtained that the set of candidates to be optimal solutions is $EQ \times Y \cup T$, that means, the points generated by the distances between each node and each equilibrium point and the set T. It should be noted that in this case we have added these points because there may exist ties which do not allow to move the service facility improving the objective function. In the 3-facility ordered median problem, the previous candidate set is not enough because if $x_1 \in EQ$ and $x_2 \in Y \setminus EQ$, the distances between each node and x_2 don't have to be included in the set of radius, R. Therefore, it may occur that there exists a tie between two nodes and the service facilities x_2 and x_3 respectively, so that there is no movement of the facilities at x_2 and x_3 which improves the objective function (see Example 4.1).

Example 4.1. Let $N = (G, l)$ be a network with underlying graph $G = (V, E)$ where $V = \{v_1, v_2, v_3, v_4, v_5, v_6\}$ and $E = \{[v_1, v_2], [v_2, v_3], [v_3, v_4], [v_4, v_5], [v_5, v_6]\}$. The length function is given by $l([v_1, v_2]) = 3, l([v_2, v_3]) = 50, l([v_3, v_4]) = 6, l([v_4, v_5]) = 50, l([v_5, v_6]) = 10$. The w-weights are all equal to one and the λ-modeling weights are $\lambda_1 = 0.1, \lambda_2 = 0.2, \lambda_3 = 0.4, \lambda_4 = 0.3, \lambda_5 = 0.6, \lambda_6 = 0.55$, see Figure 3 (in this figure we use the same notation used in Figure 1).

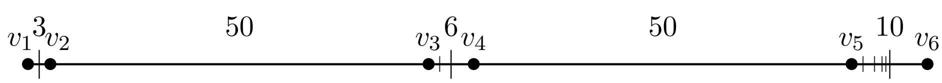

Figure 3: *Illustration of Example 4.1*

In this example the optimal solution is given by $x_1 = p([v_1, v_2], 1.5)$, $x_2 = p([v_3, v_4], 1.5)$ and $x_3 = p([v_4, v_5], 4.5)$ (see Table 3). It can be seen that x_1 is an equilibrium point, $x_2 \in Y(1.5)$ and x_3 neither belongs to Y nor is a component of a pair of T.

This example illustrates that in order to obtain the optimal solution for the 3-facility problem new points have to be added. Our conjecture is that these points can be generated using recursively the construction of the set of radii but now regarding the distances from the points in $\pi_2(F) := \{x_2 : (x_1, x_2) \in F\}$, that is, the points in $A(G)$ which correspond to the second candidate of any pair in F, and the node set:

$$R_1 = \{r : r = w_i d(v_i, y), v_i \in V, y \in \pi_2(F)\},$$

Candidate pair X_3	Val.	Candidate pair X_3	Val.
$p([v_1,v_2],0),p([v_3,v_4],0),p([v_4,v_5],0)$	10	$p([v_1,v_2],1.5),p([v_3,v_4],0),p([v_4,v_5],0)$	10.1
$p([v_1,v_2],0),p([v_3,v_4],0),p([v_4,v_5],1.5)$	9.77	$p([v_1,v_2],1.5),p([v_3,v_4],0),p([v_4,v_5],1.5)$	9.62
$p([v_1,v_2],0),p([v_3,v_4],0),p([v_4,v_5],3)$	9.55	$p([v_1,v_2],1.5),p([v_3,v_4],0),p([v_4,v_5],3)$	9.25
$p([v_1,v_2],0),p([v_3,v_4],0),p([v_4,v_5],4)$	9.3	$p([v_1,v_2],1.5),p([v_3,v_4],0),p([v_4,v_5],4)$	9
$p([v_1,v_2],0),p([v_3,v_4],0),p([v_4,v_5],4.5)$	9.15	$p([v_1,v_2],1.5),p([v_3,v_4],0),p([v_4,v_5],4.5)$	8.85
$p([v_1,v_2],0),p([v_3,v_4],0),p([v_4,v_5],5)$	9	$p([v_1,v_2],1.5),p([v_3,v_4],0),p([v_4,v_5],5)$	8.75
$p([v_1,v_2],0),p([v_3,v_4],1.5),p([v_4,v_5],0)$	9.7	$p([v_1,v_2],1.5),p([v_3,v_4],1.5),p([v_4,v_5],0)$	9.55
$p([v_1,v_2],0),p([v_3,v_4],1.5),p([v_4,v_5],1.5)$	9.17	$p([v_1,v_2],1.5),p([v_3,v_4],1.5),p([v_4,v_5],1.5)$	8.87
$p([v_1,v_2],0),p([v_3,v_4],1.5),p([v_4,v_5],3)$	8.95	$p([v_1,v_2],1.5),p([v_3,v_4],1.5),p([v_4,v_5],3)$	8.5
$p([v_1,v_2],0),p([v_3,v_4],1.5),p([v_4,v_5],4)$	8.7	$p([v_1,v_2],1.5),p([v_3,v_4],1.5),p([v_4,v_5],4)$	8.25
$p([v_1,v_2],0),p([v_3,v_4],1.5),p([v_4,v_5],4.5)$	8.57	$p([v_1,v_2],1.5),p([v_3,v_4],1.5),p([v_4,v_5],4.5)$	8.12
$p([v_1,v_2],0),p([v_3,v_4],1.5),p([v_4,v_5],5)$	8.6	$p([v_1,v_2],1.5),p([v_3,v_4],1.5),p([v_4,v_5],5)$	8.15
$p([v_1,v_2],0),p([v_3,v_4],3),p([v_4,v_5],0)$	11.2	$p([v_1,v_2],1.5),p([v_3,v_4],3),p([v_4,v_5],0)$	9.1
$p([v_1,v_2],0),p([v_3,v_4],3),p([v_4,v_5],1.5)$	8.87	$p([v_1,v_2],1.5),p([v_3,v_4],3),p([v_4,v_5],1.5)$	8.42
$p([v_1,v_2],0),p([v_3,v_4],3),p([v_4,v_5],3)$	8.35	$p([v_1,v_2],1.5),p([v_3,v_4],3),p([v_4,v_5],3)$	8.2
$p([v_1,v_2],0),p([v_3,v_4],3),p([v_4,v_5],4)$	8.4	$p([v_1,v_2],1.5),p([v_3,v_4],3),p([v_4,v_5],4)$	8.25
$p([v_1,v_2],0),p([v_3,v_4],3),p([v_4,v_5],4.5)$	8.42	$p([v_1,v_2],1.5),p([v_3,v_4],3),p([v_4,v_5],4.5)$	8.27
$p([v_1,v_2],0),p([v_3,v_4],3),p([v_4,v_5],5)$	8.45	$p([v_1,v_2],1.5),p([v_3,v_4],3),p([v_4,v_5],5)$	8.3

Table 3: *Evaluation of the candidate solutions of Example 4.1.*

$$\begin{aligned} Y_1(r) &= \{y : y \in A(G), w_i d(v_i, y) = r, v_i \in V\}, \\ Y_1 &= \bigcup_{r \in R_1} Y_1(r). \end{aligned}$$

The same situation occurs in the p-facility case, so that in general this construction must be repeated p-times in order to obtain a finite candidate set to be optimal solutions for that problem. Therefore the structure of the candidate set defined in the previous section depends on the number of facilities to be located. Hence, we conjecture that there exists no candidate set of points to be optimal solutions of Problem (1.3) with polynomial cardinality.

5 Conclusions

In this paper we have characterized a finite set of candidates to be optimal solutions for the 2-facility ordered median problem with cardinality $O(m^3n^6)$. Although the cardinality of this set is larger than the cardinality of the FDS for all the classical location problems, this set allows us to solve problems that cannot be solved with any other formulation. The main difference of the set F in (3.1) with respect to previously known FDS for other problems is that it is not valid for $p > 2$.

In fact, we show in Section 4 that the structure of the candidate set depends on the number of facilities to be located. These results shed light on the validity of general finite dominating sets for the p-facility ordered median problem. Nevertheless, it is still an open line of research whether there exists polynomial cardinality FDS for the ordered p-median problem when no hypothesis are made on the set of λ-weights.

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